Privacy

Differential Privacy

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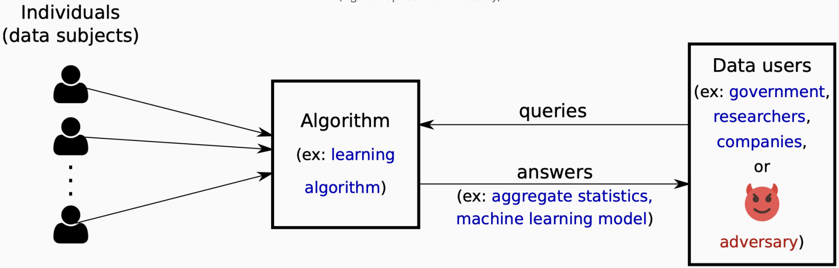
Last update: January 24, 2025

original slides from A. Bellet (Inria), M2DS Univ. Lilles
and A. Machanavajjhala, M. Hay, X. He; Differential Privacy in the Wild, VLDB'16 & SIGMOD'17 Tutorial

(Figure inspired from R. Bassily)



Goal: **achieve utility** while **preserving privacy** (conflicting objectives!)

REMINDER: REQUIREMENTS FOR PRIVACY DEFINITION

1. **Robustness to any auxiliary knowledge** the adversary may have, since one cannot predict what an adversary knows or might know in the future
2. **Composition over multiple analyses**: keep track of the “privacy budget” when asking several questions about the same data

OUTLINE

Next Topic

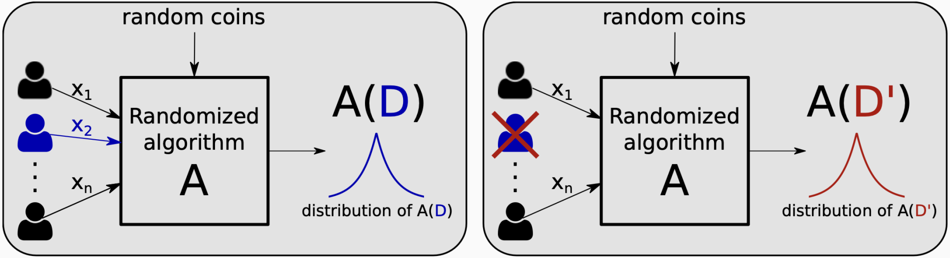
Differential Privacy (DP)

A First DP Algorithm

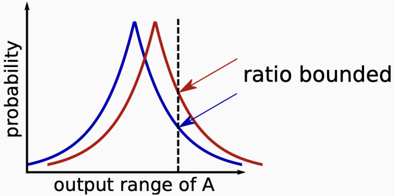
Properties of DP

SCHEMATIC DIFFERENTIAL PRIVACY

(Figure inspired from R. Bassily)



Requirement: $\mathcal{A}(D)$ and $\mathcal{A}(D')$ should have "close" distributions



Definition (Differential Privacy)

A randomized mechanism $\mathcal{A}$ preserves differential privacy if for any pair of neighboring datasets D and D', and for all possible sets of outputs S:

$$\Pr[\mathcal{A}(D) \in S] \leq e^{\varepsilon} \Pr[\mathcal{A}(D') \in S], \quad \varepsilon > 0$$

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First introduced in [Dwork et al., 2006] by **Dwork, McSherry, Nissim and Smith who won the Gödel prize in 2017**

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- Why all pairs of datasets?
  - Privacy guarantee holds no matter what the other records are
- Why all outputs?
  - Should not be able to distinguish whether input was D or D' no matter what the output

ABOUT ε PARAMETER

Privacy budget is actually a privacy loss

$$\ln\left(\frac{\Pr[\mathcal{A}(D) \in S]}{\Pr[\mathcal{A}(D') \in S]}\right) \leq \varepsilon$$

Small value of ε requires $\mathcal{A}$ to provide very similar outputs when given similar inputs

How should we set ε to prevent bad outcomes in practice? Nobody knows...

- Remind $e^{\varepsilon} \approx 1 + \varepsilon$ for very small ε values

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- up to 1.0 gives a strong privacy: $\varepsilon = 0.1$ bounds leak to 10%
- 1.0 to 10 is "better than nothing"
- more than 10 hardly protects privacy...

WHY S IS A SET?

$\mathcal{A}(D) \in S$ vs. $\mathcal{A}(D) = s$?

If $\mathcal{A}$ returns elements from a continuous output domain, $\Pr[\mathcal{A}(D) = s] = 0$ for all D

The DP definition makes sense for both discrete and continuous distributions.

For discrete outputs, then the definition may be

$$\Pr[\mathcal{A}(D) = s] \leq e^{\varepsilon} \Pr[\mathcal{A}(D') = s]$$

CAN DETERMINISTIC ALGORITHMS SATISFY DP?

Non-trivial deterministic algorithm has at least two distinct outputs in its image

There exist two inputs that differ in one row, mapped to distinct outputs:

- Assume $D = D' \cup \{x\}$, x the target row,
- and $\mathcal{A}(D) = o_1$, $\mathcal{A}(D') = o_2$ deterministically (so undoubtedly)

Then, a Differencing Attack may disclose the target's data

Aside, $\Pr[\mathcal{A}(D) = o_1] = 1.0$ and $\Pr[\mathcal{A}(D') = o_1] = 0.0$

WHAT ABOUT RANDOM SAMPLING?

Assume $D = D' \cup \{x\}$, x the target row;

As soon as row x is sampled in o, then $\Pr[\mathcal{A}(D') = o] = 0.0$, and

$$\frac{\Pr[\mathcal{A}(D) \in S]}{\Pr[\mathcal{A}(D') \in S]} = +\infty$$

Privacy loss is infinite!

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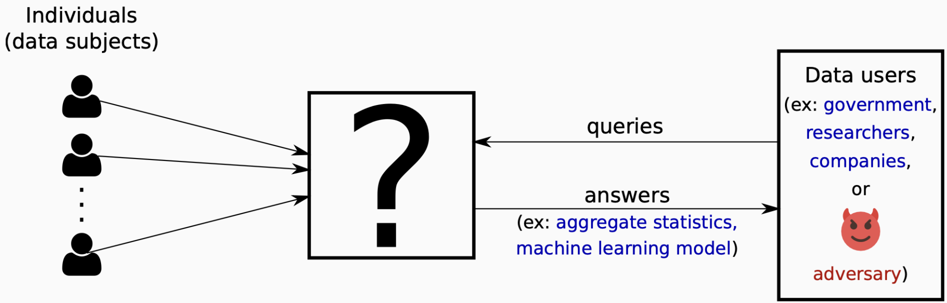
HOW TO DESIGN DP ALGORITHMS?



ANSWERING NUMERICAL QUERIES

- Suppose we want to compute a numerical function $f : \mathcal{X}^n \to \mathbb{R}$ of a private dataset D
- How to construct a DP algorithm (or mechanism $\mathcal{A}$) for computing $f(D)$?
  - How much randomness (error) do we add?
  - How to introduce this randomness in the output?

A popular approach: the Laplace mechanism

Algorithm: Laplace mechanism $\mathcal{A}_{\text{Lap}}(D, f : \mathcal{X}^n \to \mathbb{R}, \varepsilon)$

1. Compute $\boldsymbol{\Delta} = \boldsymbol{\Delta}_1(f)$, the sensitivity of function f
2. draw $Y \sim \text{Lap}(\boldsymbol{\Delta}/\varepsilon)$, the added noise
3. Output $f(D) + Y$, the noisy answer to query f over D

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Idea

perturb $f(D)$ with Laplace noise, to get $\mathcal{A}_{\text{Lap}}(D, f, \varepsilon) := f(D) + \text{Lap}(\frac{\boldsymbol{\Delta}}{\varepsilon})$

- noise is calibrated to sensitivity $\boldsymbol{\Delta}$ of f and the privacy parameter ε

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Theorem (DP guarantees for Laplace mechanism)

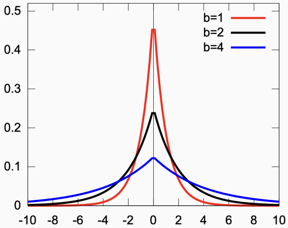
The Laplace mechanism $\mathcal{A}_{\text{Lap}}(D, f, \varepsilon)$ *satisfies* ε*-differential privacy*

Definition (Laplace distribution)

The Laplace distribution Lap(b) (centered at 0) with scale b is the distribution with probability density function:

$$p(y; b) = \frac{1}{2b} \exp\left(-\frac{|y|}{b}\right), \quad y \in \mathbb{R}.$$

- It is a symmetric version of the exponential distribution
- For $Y \sim$ Lap(b), we have $\mathbb{E}[Y] = 0$, $\mathbb{E}[|Y|] = b$, $\mathbb{E}[Y^2] = 2b^2$
- Useful property for DP: Pr$[Y = y]$/Pr$[Y + a = y]$ can be bounded by something which does not depend on y



- This is great but what is the error incurred when using $\mathcal{A}_{\text{Lap}}(D, f, \varepsilon)$ to answer $f(D)$?
- For a given output of $\mathcal{A}_{\text{Lap}}(D, f, \varepsilon)$, we can consider the ℓ_1 error $||\mathcal{A}_{\text{Lap}}(D, f, \varepsilon) - f(D)||_1$

Theorem (Expected ℓ_1 error of the Laplace mechanism)

Let $\varepsilon > 0$. For a query $f : \mathcal{X}^n \to \mathbb{R}$ and any dataset $D \in \mathcal{X}^n$, the Laplace mechanism $\mathcal{A}_{\text{Lap}}(D, f, \varepsilon)$ has the following utility guarantee:

$$\mathbb{E}[||\mathcal{A}_{\text{Lap}}(D, f, \varepsilon) - f(D)||_1] = \frac{\boldsymbol{\Delta}_1(f)}{\varepsilon}.$$

- The Laplace mechanism can answer low sensitivity queries, say $\boldsymbol{\Delta}_1(f) = O(1)$ or smaller, with high utility (as long as ε is not too small)

THE LAPLACE MECHANISM: USE CASE

- Assume $\boldsymbol{\Delta}_1(f) = 1$ and $\varepsilon = 0.1$
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- Let's compute the probability of the "tail region", i.e. noise $> b$:

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\begin{aligned}
2 \cdot \int_b^{\infty} p(y; b)\, dy \quad &= \quad 2 \cdot \frac{1}{2b} \cdot \int_b^{\infty} \exp\left(-\frac{|y|}{b}\right) dy \\
&= \quad -\frac{2b}{2b} \cdot \left[e^{-\frac{y}{b}}\right]_b^{\infty} \quad = \quad e^{-1} \quad = \quad 0.36
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- Is this answer useful?
    - Yes, if the real answer is $\gg$ 10
    - No, otherwise

Definition (Global ℓ_1 sensitivity)

The global ℓ_1 sensitivity of a query (function) $f : \mathcal{X}^n \to \mathbb{R}$ is

$$\boldsymbol{\Delta}_1(f) = \max_{D,D':D\Delta D' \leq 1} |f(D) - f(D')|_1$$

- *global* means it holds for all pairs of neighboring datasets
- How much one record can affect the output value of the function
- Intuitively, it gives the amount of uncertainty needed to hide any single contribution

INTERPRETING GLOBAL SENSITIVITY

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- What is the sum of the salaries, knowing salaries range between 20K€ and 200K€
- What is the average age?

CLIPPING

Queries with unbounded sensitivity cannot be straightforwardly answered with the Laplace mechanism

Definition (Clipping)

Enforce lower and upper bounds of a given function, as a *band-pass filter*, to fall back into bounded sensitivity

- Trade-off between information lost in clipping and noise needed to ensure DP
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Sensitivity underestimation may break the differential privacy guarantee, while sensitivity overestimation leads to unnecessary inaccuracy in the private analysis

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ROBUSTNESS TO AUXILIARY KNOWLEDGE

- DP guarantees are intrinsically robust to arbitrary auxiliary knowledge: it bounds the relative advantage that an adversary gets from observing the output of an algorithm
    - Adversary may know all the dataset except one record
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  - Adversary may know all the dataset except one record
  - Adversary may know all external sources of knowledge, present and future
- The algorithm $\mathcal{A}$ can be public: only the randomness needs to remain hidden
  - A key requirement of modern security ("security by obscurity" has long been rejected)
  - Allows to openly discuss the algorithms and their guarantees

Theorem (Postprocessing)

Let $\mathcal{A} : \mathcal{X}^n \to \mathcal{O}$ be ε-DP and let $f : \mathcal{O} \to \mathcal{O}'$ be an arbitrary (randomized) function, independent of A. Then

$$f \circ \mathcal{A} : \mathcal{X}^n \to \mathcal{O}'$$

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- "Thinking about" the output of a differentially private algorithm cannot make it less differentially private
- Can let data users do whatever they want with it
- This holds regardless of attacker strategy and computational power

Theorem (Simple composition)

Let $\mathcal{A}_1, \ldots, \mathcal{A}_K$ be K independently chosen algorithms where $\mathcal{A}_k$ satisfies ε_k-DP. For any dataset D, let $\mathcal{A}$ be such that

$$\mathcal{A}(D) = (\mathcal{A}_1(D), \ldots, \mathcal{A}_K(D)).$$

Then $\mathcal{A}$ is ε-DP with $\varepsilon = \sum_{k=1}^{K} \varepsilon_k$.

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Then $\mathcal{A}$ is ε-DP with $\varepsilon = \sum_{k=1}^{K} \varepsilon_k$.

- This allows to control the cumulative privacy loss over multiple analyses run on the same dataset, including complex multi-step algorithms
- Total budget is an upper bound: actual privacy loss may be smaller
  - $(\text{Lap}(1/\varepsilon_1) + \text{Lap}(1/\varepsilon_2))/2$ is less accurate than $\text{Lap}(1/(\varepsilon_1 + \varepsilon_2))$

The previous composition result is worst-case (assumes correlated outputs)

Theorem (Parallel composition)

If $\mathcal{A}_1, \ldots, \mathcal{A}_K$ operate on distinct inputs, then $\mathcal{A}(D)$ is $\max_k \varepsilon_k$-DP

Example (Count by gender and hair color)

	Blond	Dark	Brown	Red
Female	20	33	9	7
Nonbinary	12	7	28	3
Male	17	42	4	8

If for each count the algorithm generating it satisfies ε-DP, then releasing the entire table is also ε-DP (as opposed to 12ε-DP with sequential composition!)

CONCLUSION

- Differential Privacy is robust to auxiliary knowledge
- DP is a property of the algorithm, not the dataset
- DP requires randomization
- Privacy loss is bounded by ε, also called “budget”
- The Laplace Mechanism provides ε-DP to numerical functions (queries)
- Laplace scale is calibrated to sensitivity of the function and ε
- Clipping ensures sensitivity is bounded
- DP mechanisms can be composed
    - in sequence, then $\varepsilon = \sum \varepsilon_k$, or
    - in parallel, then $\varepsilon = \max \varepsilon_k$
- DP is robust to postprocessing

Dwork, C., McSherry, F., Nissim, K., and Smith, A. (2006).
Calibrating noise to sensitivity in private data analysis.
In *Proceedings of the Third Conference on Theory of Cryptography*, TCC'06, page 265–284, Berlin, Heidelberg. Springer-Verlag.