Bayesian Networks - I : Definition and probabilistic inference

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Reminders of basic probabilistic theory

Conditional probability

- let A and M denote two events
- a priori information about A : $P(A)$
- M happened : $P(M) \neq 0$
- if there is a link between A and M, this event will modify our knowledge about A
- a posteriori information : $P(A|M) = \frac{P(A,M)}{P(M)}$

Reminders of basic probabilistic theory

Independence

- A and B are independent iff :
  $P(A, B) = P(A) \times P(B)$
  $P(A|B) = P(A)$
  $P(B|A) = P(B)$

Conditional independence

- A and B are independent conditionally to C iff :
  $P(A|B, C) = P(A|C)$

Reminders of basic probabilistic theory

$\{M_i\}$ complete set of mutually exclusive events

Marginalization : $P(A) = \sum_i P(A, M_i)$

Total probability theorem

Event A can result from various causes M_i. What is the probability of A if we know :

- *the prior probabilities $P(Mi)$*
- *the conditional probabilities of A given each Mi*

$$P(A) = \sum_i P(A|M_i)P(M_i)$$

Reminders of basic probabilistic theory

$\{M_i\}$ complete set of mutually exclusive events

Bayes' theorem

Event A happened. What is the probability that the cause M_i is responsible of this event ?

$$P(M_i|A) = \frac{P(A|M_i) \times P(M_i)}{P(A)}$$

- $P(Mi|A)$: a posteriori probability
- $P(A)$: constant w.r.t. Mi (cf. Total probability theorem)

Chain rule

$$P(A_1 \ldots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2) \ldots P(A_n|A_1 \ldots A_{n-1})$$

Example

Bayesian network definition

Theoretical principle

- taking into account some extra knowledge (conditional independence between some variables) to simplify the joint probability distribution given by the chain rule.

Definition [Pearl, 1985]

- a Bayesian network (BN) is defined by
  - one qualitative description of (conditional) dependences/independences between variables
    directed acyclic graph (DAG)
  - one quantitative description of these dependences
    conditional probability distributions (CPDs)

Example

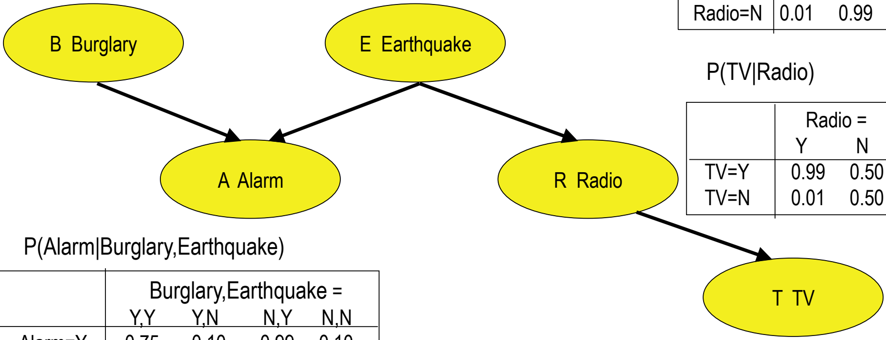
one topological order : B, E, A, R, T (not unique)

P(Burglary)=[0.001 0.999]

P(Earthquake)=[0.0001 0.9999]

P(Radio|Earthquake)

	Earthquake = Y	N
Radio=Y	0.99	0.01
Radio=N	0.01	0.99



P(TV|Radio)

	Radio = Y	N
TV=Y	0.99	0.50
TV=N	0.01	0.50

P(Alarm|Burglary,Earthquake)

	Burglary,Earthquake = Y,Y	Y,N	N,Y	N,N
Alarm=Y	0.75	0.10	0.99	0.10
Alarm=N	0.25	0.90	0.01	0.90

BN as a dependence model

Dependence is a symmetrical relationship, so why using directed edges ?

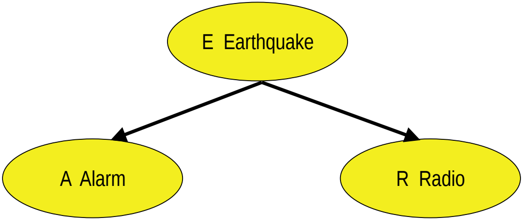
Example with 3 nodes

- 3 simple structures between A, B and C :
  - $A \rightarrow C \rightarrow B$: serial connexion
  - $A \leftarrow C \rightarrow B$: divergent connexion
  - $A \rightarrow C \leftarrow B$: convergent connexion (V-structure)

Serial connexion

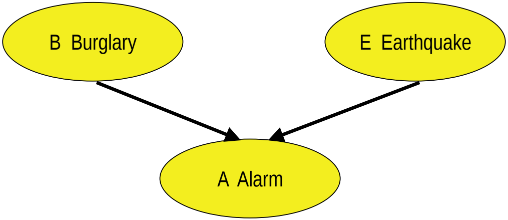
- E and T are dependent
- E and T are independent conditionally to R
  - if R is known, T will not give any new information about E
  - $P(T|E,R) = P(T|R) = P(T|parents(T))$

Divergent connexion



- A and R are dependent
- A and R are independent conditionally to E
  - if E is known, A will not give any new information about R
  - $P(R|A,E) = P(R|E) = P(R|parents(R))$

Convergent connexion – V-structure



- B and E are independent
- B and E are dependent conditionally to A
  - if A is known, E will give some new information about B
  - $P(A|B, E) = P(A|parents(A))$

Consequence

Chain rule

$$P(S) = P(S_1) \times P(S_2|S_1) \times P(S_3|S_1, S_2) \times \cdots \times P(S_n|S_1 \dots S_{n-1})$$

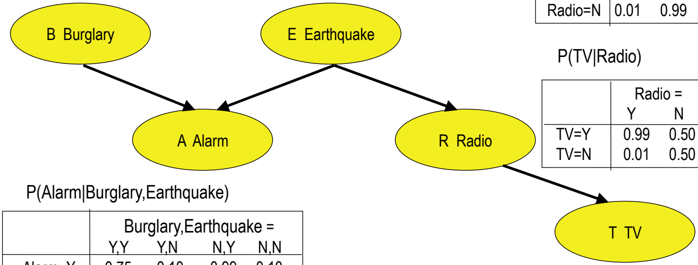
Consequence with a BN

- $P(S_i|S_1 \dots S_{i-1}) = P(S_i|parents(S_i))$ so
$$P(S) = \Pi_{i=1}^{n} P(S_i|parents(S_i))$$
- the (global) joint probability distribution is decomposed in a product of (local) conditional distributions
- BN = compact representation of the joint distribution $P(S)$ given some information about dependence relationships between variables

Example

P(Burglary)=[0.001 0.999]

P(Earthquake)=[0.0001 0.9999]



P(Radio|Earthquake)

	Earthquake = Y	N
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P(TV|Radio)

	Radio = Y	N
TV=Y	0.99	0.50
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P(Alarm|Burglary,Earthquake)

	Burglary,Earthquake = Y,Y	Y,N	N,Y	N,N
Alarm=Y	0.75	0.10	0.99	0.10
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$$P(B, E, A, R, T) =$$
$$P(B) \times P(E|B) \times P(A|B, E) \times P(R|B, E, A) \times P(T|B, E, A, R)$$
$$P(B) \times \ P(E) \times P(A|B, E) \times \qquad P(R|E) \qquad \times P(T|R)$$

Markov equivalence

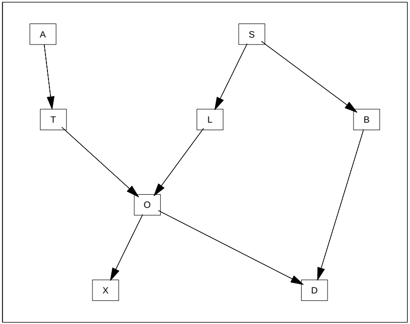
Definition

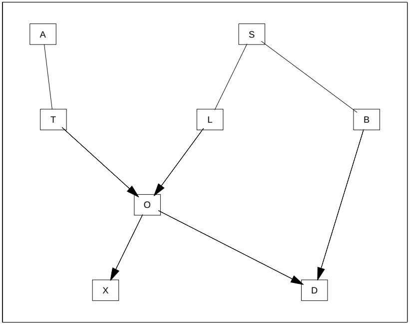
B_1 and B_2 are Markov equivalent iff both describe exactly the same conditional (in)dependence statements.

Graphical properties

- B_1 and B_2 have the same skeleton, V-structures and inferred edges.
- all the equivalent graphs (= equivalence class) can be summarized by one partially directed DAG named CPDAG or Essential Graph

Markov equivalence





Faithfulness

Definition

a Bayesian network structure G and an associated probability distribution P are faithful to one another if and only if every conditional independence relationship valid in P can be read in G

Very simple counterexample

- $G = X_1 \longrightarrow X_2$
- $P(X_2|X_1 = 0) = P(X_2|X_1 = 1) = [0.8 \;\; 0.2]$
- X_1 and X_2 are dependent in G but independent in P

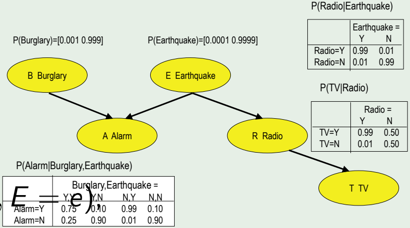
BN as a generative model

Principle

- BN = compact representation of the joint distribution $P(S)$
- we can use classical sampling methods to generate data from this distribution

Example : forward sampling

- if $rand1 < P(B = Y)$,
  $B = Y$, else N
- if $rand2 < P(E = Y)$,
  $E = Y$, else N
- if $rand3 < P(A = Y|B = b, E = e)$,
  $A = Y$, else N
- ...



Example - Minesweeper

Principle

- bombs are places in a grid
- each square (i, j) independently has a bomb ($B_{i,j} = true$) with probability b
- what you can observe for a given square is a reading $N_{i,j}$ of the number of bombs in adjacent squares (not including the square itself)

Example - Minesweeper

Bayesian network ?

- draw a Bayesian network for a one-dimensional 4x1 Minesweeper grid, showing all eight variables ($B_1 \dots B_4$ and $N_1 \dots N_4$). Show the minimal set of arcs needed to correctly model the domain above
- fully specify the CPTs for each variable, assuming that there is no noise in the readings (i.e. that the number of adjacent bombs (or bomb) is reported exactly, deterministically). Your answers may use the bomb rate b if needed
- what are the posterior probabilities of bombs B_i in each of the four squares, given no information? If we observe $N_2 = 1$, what are the posterior probabilities of bombs in each square?
- check your model with pyAgrum

What is probabilistic inference ?

Inference

- computation of any $P(S_i|\{S_j = x\})$ (NP-hard)
- evidence ε = set of observable variables $\{S_j = x\}$

Exact inference algorithms

- Message Passing (Pearl 1988) for trees or poly-trees
- Junction Tree (Jensen 1990)
- Shafer-Shenoy (1990)

Problem = combinatorial explosion for strongly connected graphs.

Approximate inference algorithms

- sampling
- variational methods

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Approximate inference algorithms

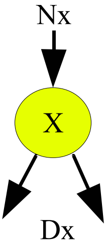
- sampling
- variational methods

Message Passing (Pearl 1988)

Principle

- designed for tree structures (generalized to poly-trees)
- every node send messages to its parent and children

- ε = set of instantiated/observed variables.
  $\varepsilon = N_x \cup D_x$ instantiated (non) descendants of X
- we can demonstrate that $P(X|\varepsilon = e) \propto \lambda(X)\pi(X)$
  with $\lambda(X) \propto P(D_x|X)$ and $\pi(X) \propto P(X|N_x)$
- 2 types of messages $\overrightarrow{\lambda}$ and $\overrightarrow{\pi}$ will help to compute these λ and π values for every X



Message Passing : $\overrightarrow{\lambda}$ messages

$\lambda(X) \propto P(D_x|X)$ information from descendants

$\lambda(X)$ initialization

- if X is an unobserved leaf : $\lambda(X) = [1 \dots 1]$ (no information)
- if X is an observed node : $\lambda(X) = [001 \dots 0]$ (exact info.) (1 at i-th position corresponds to the observed value $X = i$)

$\overrightarrow{\lambda}$ propagation and aggregation

- for every child Y of X,

$$\overrightarrow{\lambda_Y}(X = x) = \sum_y P(Y = y|X = x)\lambda(Y = y)$$

- aggregation : $\lambda(X = x) = \Pi_{Y \in Child(X)} \overrightarrow{\lambda_Y}(X = x)$

Message Passing : $\overrightarrow{\pi}$ messages

$\pi(X) \propto P(X|N_x)$ information from non descendants

$\pi(X)$ initialization

- if X is the unobserved root : $\pi(X) = P(X)$ (a priori info.)
- if X is an observed node : $\pi(X) = [001\ldots0]$ (exact info.)

$\overrightarrow{\pi}$ propagation and aggregation

- for Z, unique parent of X,

$$\overrightarrow{\pi_X}(Z = z) = \pi(Z = z) \prod_{U \in Child(Z) \setminus \{X\}} \overrightarrow{\lambda_U}(Z = z)$$

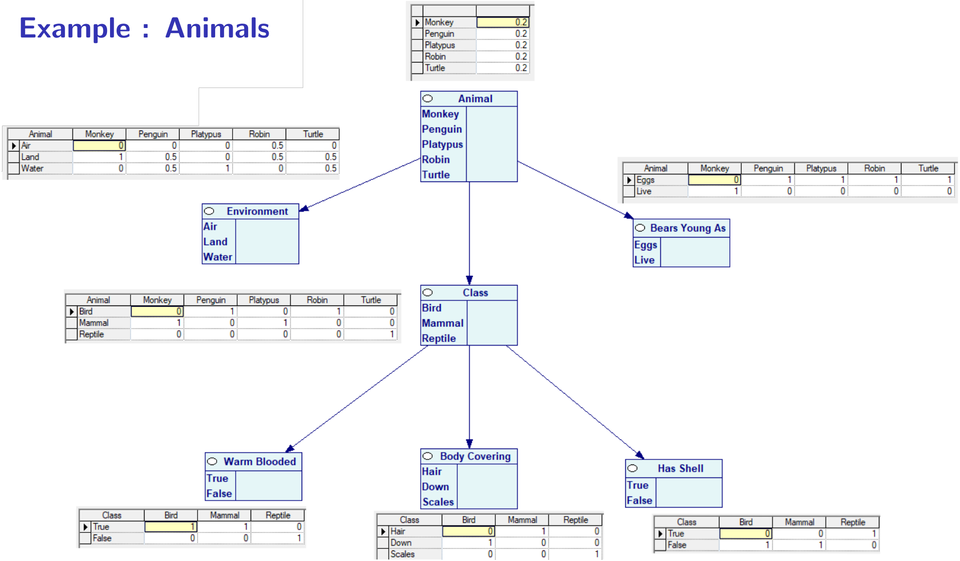
- aggregation : $\pi(X = x) = \sum_z P(X = x|Z = z)\overrightarrow{\pi_X}(Z = z)$

Message Passing complexity

Worst case

- each node send 2 messages : time complexity is linear in the number of nodes
- work done in a node is proportional to the size of its CPD : linear for trees, but need to be bounded for poly-trees

Example : Animals



Monkey	0.2
Penguin	0.2
Platypus	0.2
Robin	0.2
Turtle	0.2

Animal	Monkey	Penguin	Platypus	Robin	Turtle
Air	0	0	0	0.5	0
Land	1	0.5	0	0.5	0.5
Water	0	0.5	1	0	0.5

Animal	Monkey	Penguin	Platypus	Robin	Turtle
Eggs	0	1	1	1	1
Live	1	0	0	0	0

Animal	Monkey	Penguin	Platypus	Robin	Turtle
Bird	0	1	0	1	0
Mammal	1	0	1	0	0
Reptile	0	0	0	0	1

Class	Bird	Mammal	Reptile
True	1	1	0
False	0	0	1

Class	Bird	Mammal	Reptile
Hair	0	1	0
Down	1	0	0
Scales	0	0	1

Class	Bird	Mammal	Reptile
True	0	0	1
False	1	1	0