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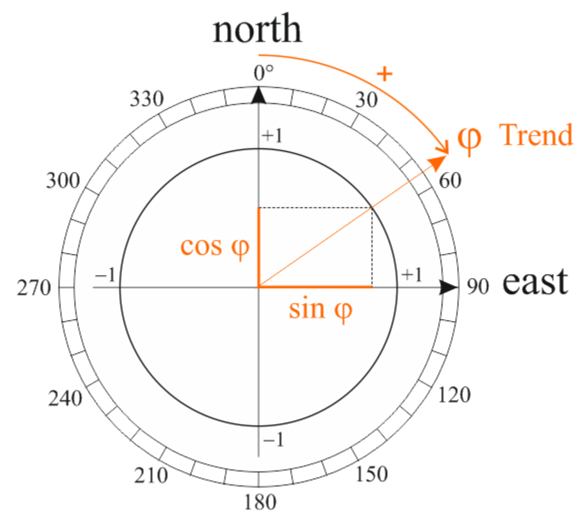
The orientation convention is a key issue of SPO analysis. It is the first source of error that require special attention.
We will always use the so called right hand rule in a North, East, Down clockwise orthonormal coordinate system.

In such coordinate system angles are all measured clockwise from the north from 0 to 360 degrees.

In this example the cosine direction of the **trend** direction **φ** are :

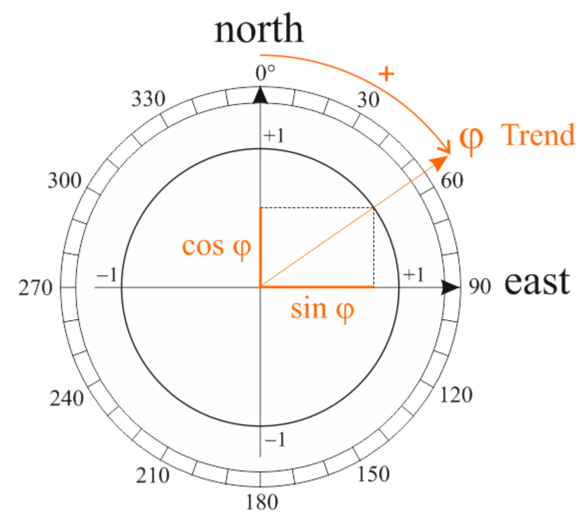
$$n_\varphi = \cos\varphi$$
$$e_\varphi = \sin\varphi$$

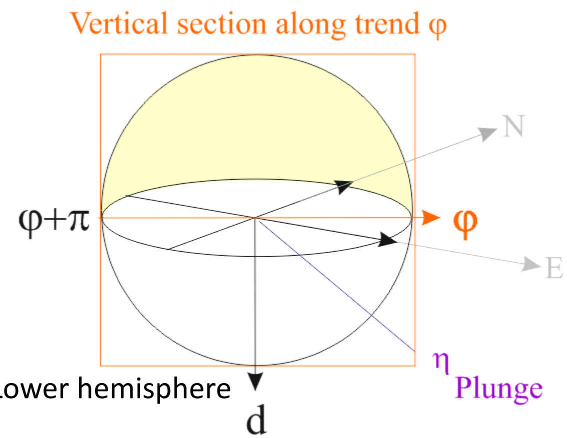


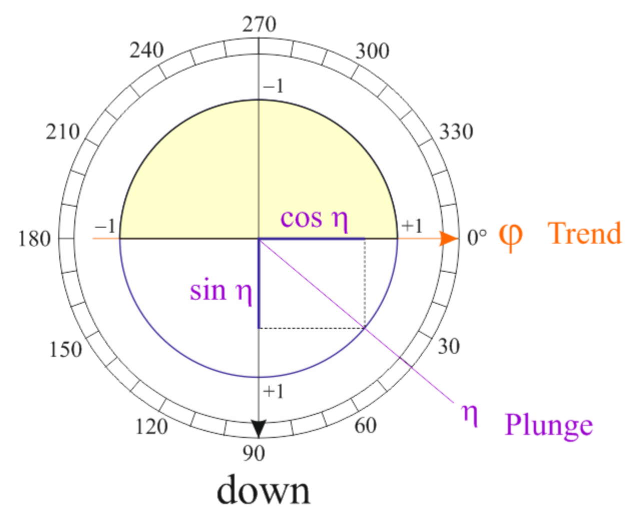
One horizontal direction measured clockwise from the North is also called **azimuth**

Let's take a η degree of a line plunging towards the trend direction φ.

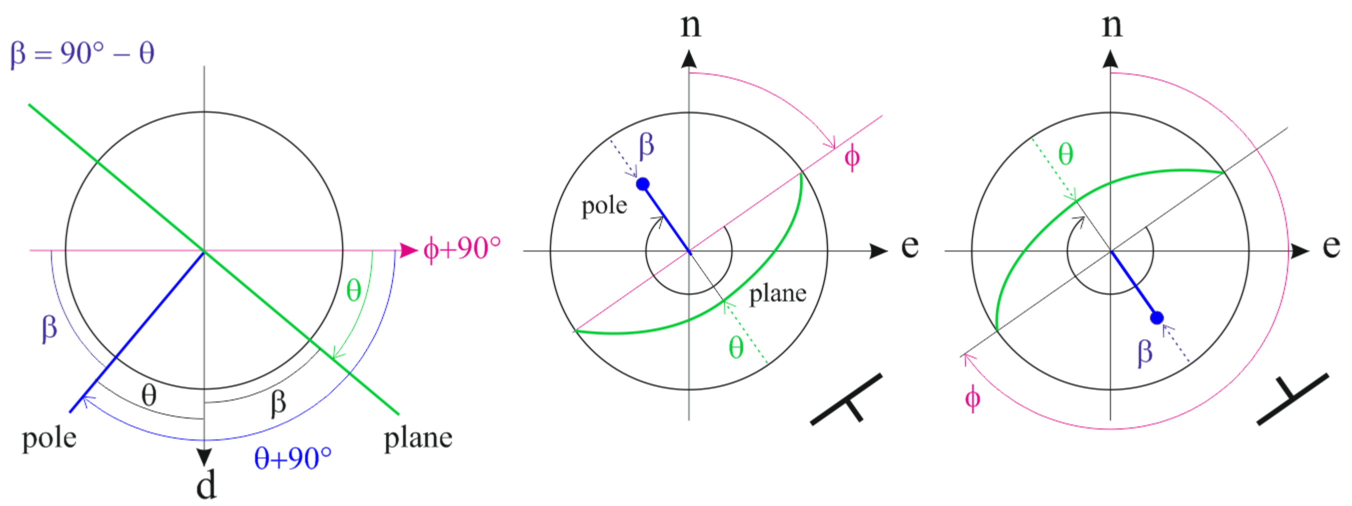
Its cosine directions on a vertical plane parallel to the **trend φ** are measured from the horizontal direction φ clockwise downwards to the **plunge η**.
In such coordinate trend plunge system angles η are measured clockwise from the azimuth direction from 0 to 180 degrees.





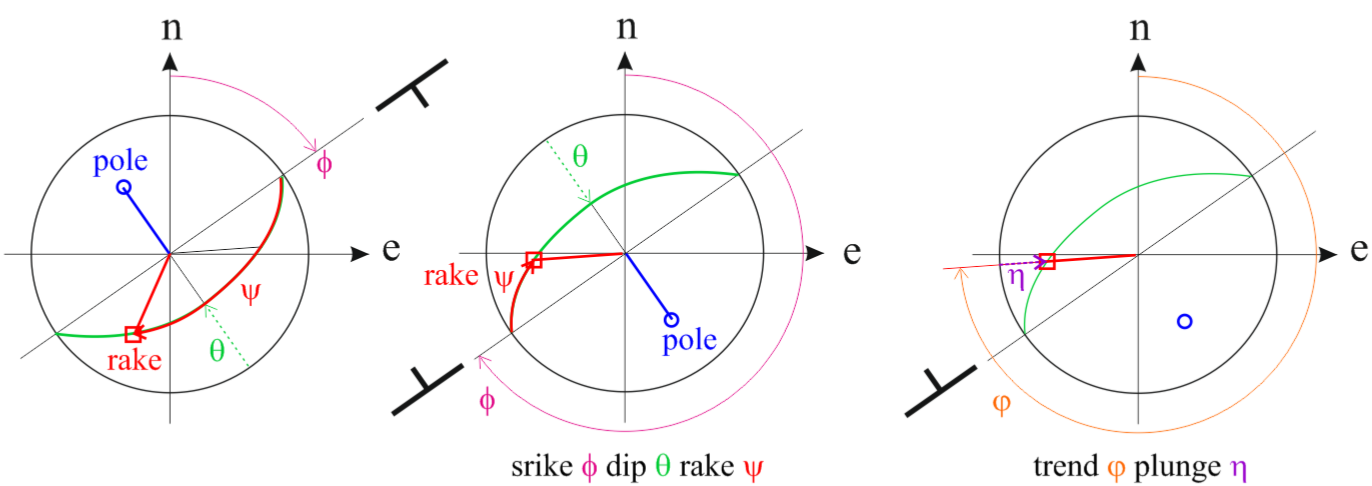


A plane cutting the horizontal along a **strike** line makes an angle ϕ with the north direction and **dip** down with an angle θ towards the clockwise perpendicular direction of the strike at $\phi + 90°$, also called right hand rule.



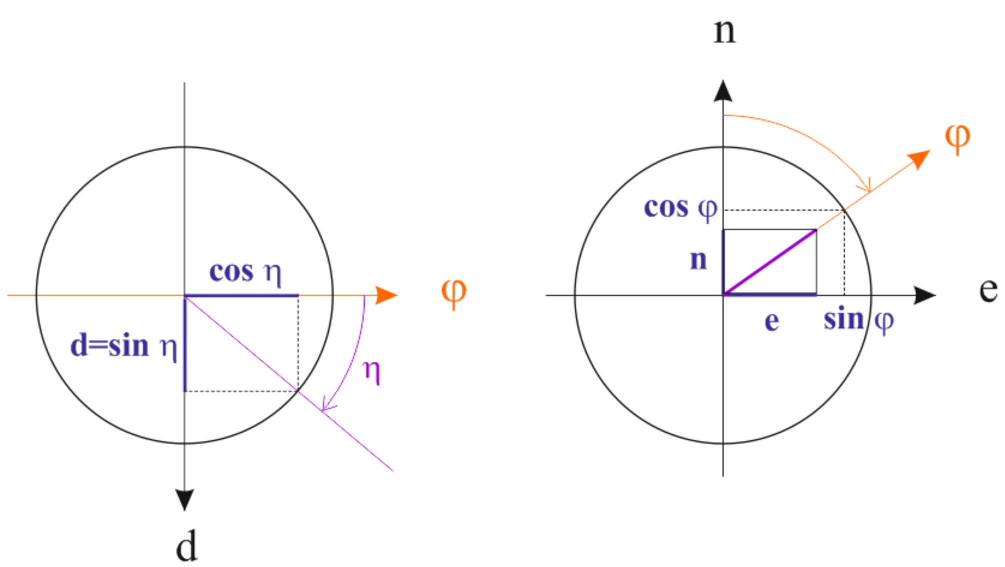
It is useful to represent a plane by its unique perpendicular line called pole of the plane. This pole is a line plunging in the opposite direction of the plane dip, always at $\theta + 90°$ of the strike perpendicular $\phi + 90°$.
The right hand rule point out the fact that the plane is always dipping in the a perpendicular direction of the strike at $\phi + 90°$ on the right of the strike direction ϕ, even when ϕ has an angle greater than $180°$ up to a close angle to $360°$.

It is also possible to measure directly a line on a plane by taking the clockwise angle **rake** or **pitch** angle ψ made between the horizontal strike direction ϕ and the line footprint on the plane just like would do a compass laid on that plane carefully aligned on the strike horizontal.



The rake ψ measured flat on the plane, from the strike ϕ horizontal direction, should not be confused with the plunge η measured on a vertical plane, from the trend φ horizontal direction, as shown in the figure above.

The orientation convention can be summarized for a line as:



Spherical coordinates:

$n = \cos\eta \cdot \cos\varphi$

$e = \cos\eta \cdot \sin\varphi$

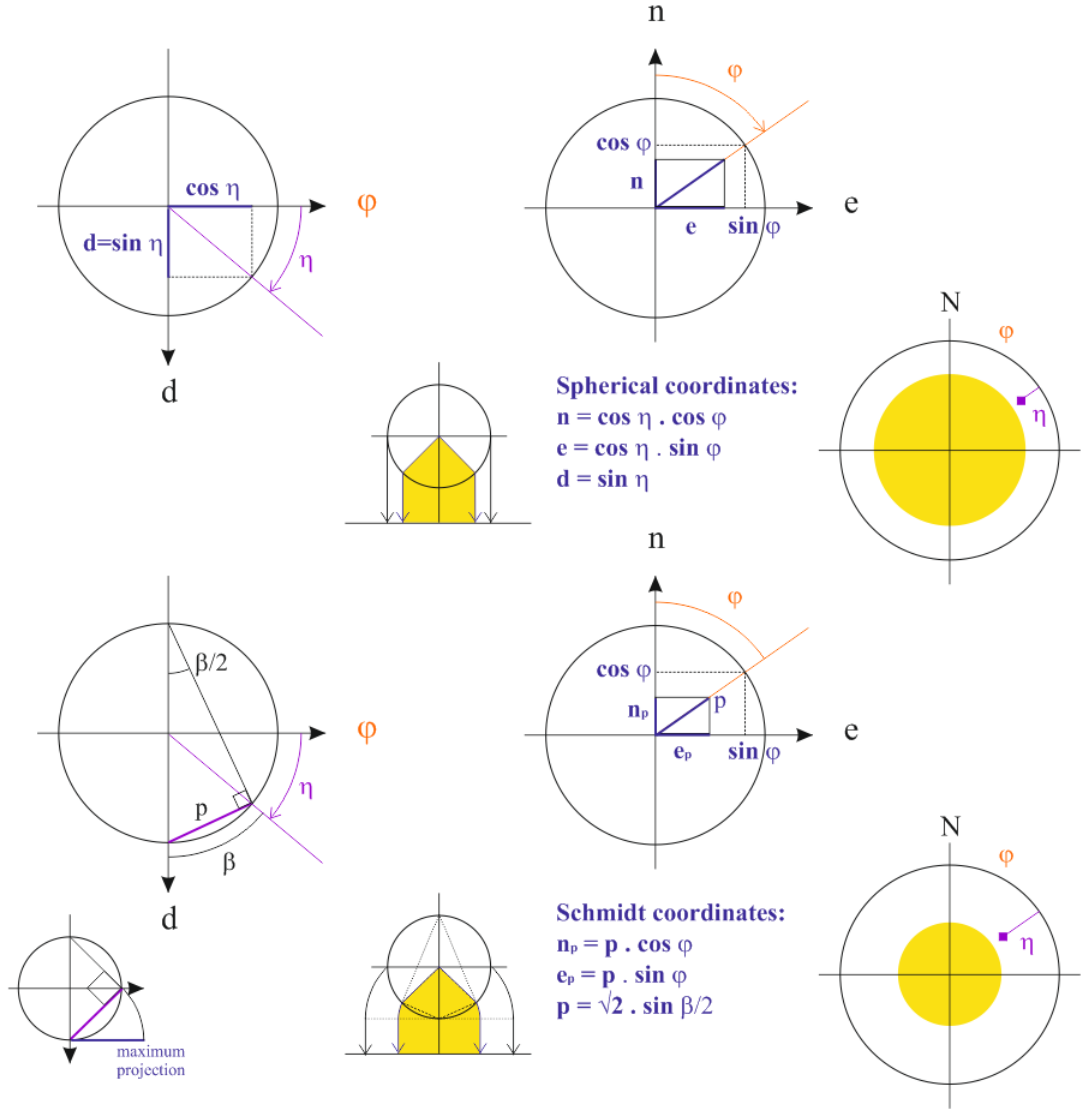
$d = \sin\eta$

In such spherical coordinates system the projection length of a line on the horizontal is the cosine of its dip angle which is used to weight its trend north and east coordinates whereas its down coordinate is simply the sinus of its plunge angle.

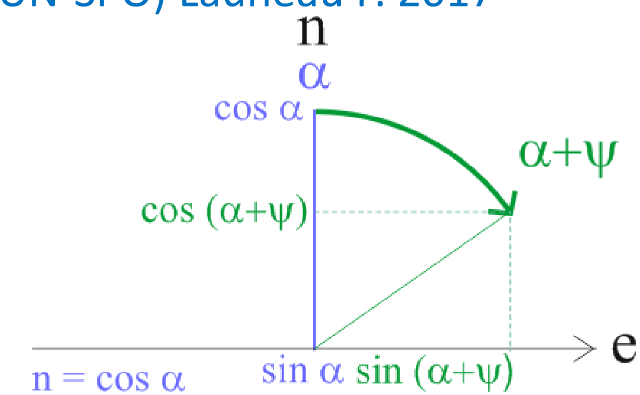
We never display directly the raw projection of the spherical coordinates in which lines dipping over 45° (yellow) occupy most of the plot area.

Like in other structural geology studies we rather use, <u>only for display</u>, the Schmidt stereonet which is a Lambert azimuthal equal-area projection. All angular cones have the same area which facilitate the visual inspection of angular distributions.

All calculations remain done in spherical coordinates.



Let α be the initial orientation of a line and $\alpha+\psi$ its final orientation after a rotation of ψ degrees.
The initial coordinates (n,e) and the final one (n’,e’) are given by the following equations which can be rearranged to write the final cosine directions (n’,e’) as function of the initial coordinates (n,e) combined with angle of rotation ψ.



$n = \cos\alpha$

$e = \sin\alpha$

$n' = \cos(\alpha+\psi) = \cos\alpha\cos\psi - \sin\alpha\sin\psi$

$e' = \sin(\alpha+\psi) = \cos\alpha\sin\psi + \sin\alpha\cos\psi$

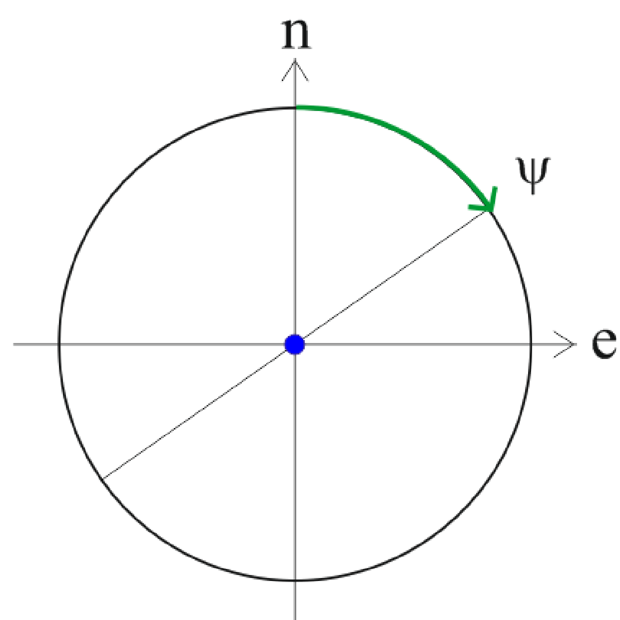
$n' = n\cos\psi - e\sin\psi$

$e' = n\sin\psi + e\cos\psi$

This can be written as a matrix of rotation as it follows in 2D:

$$\begin{vmatrix} n' \\ e' \end{vmatrix} = \begin{vmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{vmatrix} \cdot \begin{vmatrix} n \\ e \end{vmatrix}$$

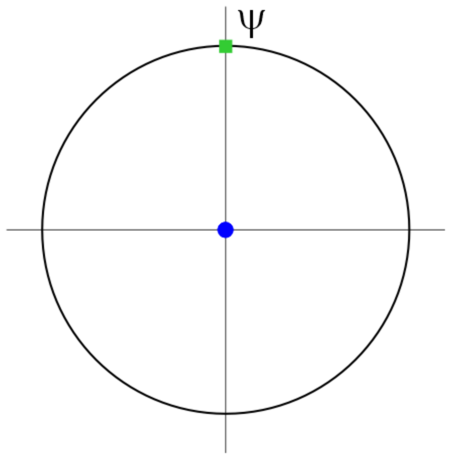
$$V' = \mathbf{R}\,V$$



The angle ψ taken from the strike of a given plane is a **pitch** or a **rake**

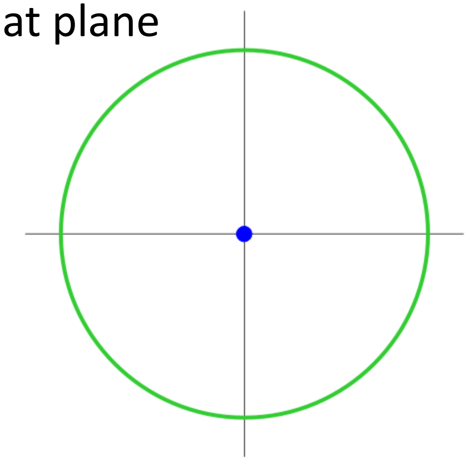
In 3D let start with one horizontal line pointing towards the north. Its spherical coordinates (n,e,b) are (1,0,0)

n	e	b
1	0	0
0	1	0
0	0	1



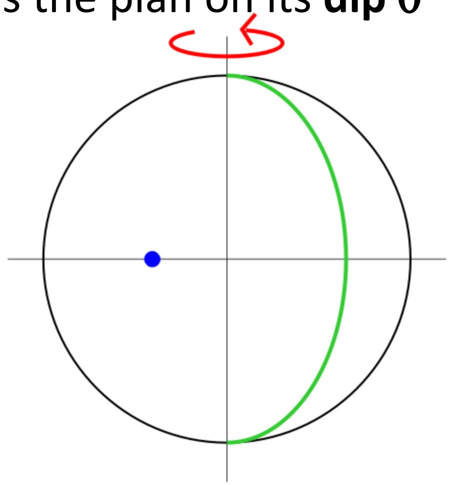
A clockwise rotation of ψ from 0° to 360° on the plane (n,e) would draw a circle around an axis of rotation (0,0,1) or eventually the **rake ψ** of any line on that plane

$$\begin{matrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{matrix}$$



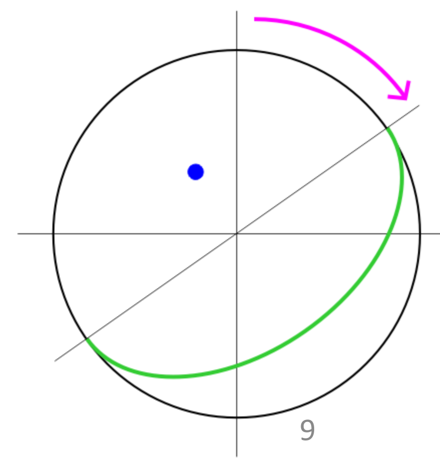
A clockwise rotation of θ around the axis (1,0,0) inclines the plan on its **dip θ**

$$\begin{matrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{matrix}$$



Finally an ultimate rotation of ϕ around (0,0,1) orients the plan along its **strike ϕ**

$$\begin{matrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{matrix}$$



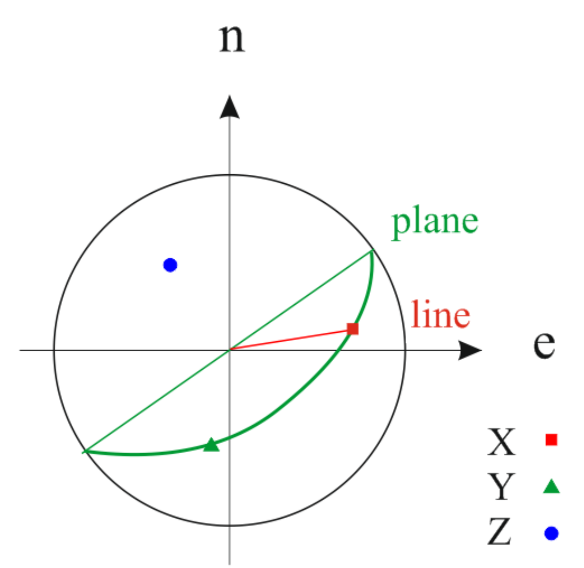
We may cumulate all strike, dip and rake rotations at once in one matrix of rotation product of the 3 individual rotations :

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

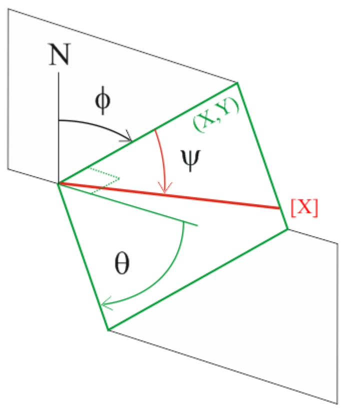
$$\begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

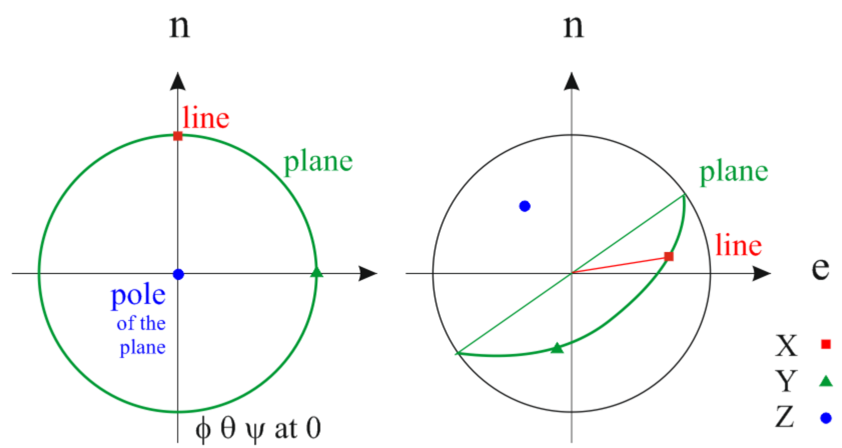
Alternatively we can use these 3 angles for the orientation of a line [X] measured on a plan (X,Y) which can be represented by its unique perpendicular pole [Z]

$$\mathbf{R} = \mathbf{R}_\phi \,.\, (\,\mathbf{R}_\theta \,.\, \mathbf{R}_\psi\,)$$



The main directions X,Y,Z can be easily retrieved with the following calculation :





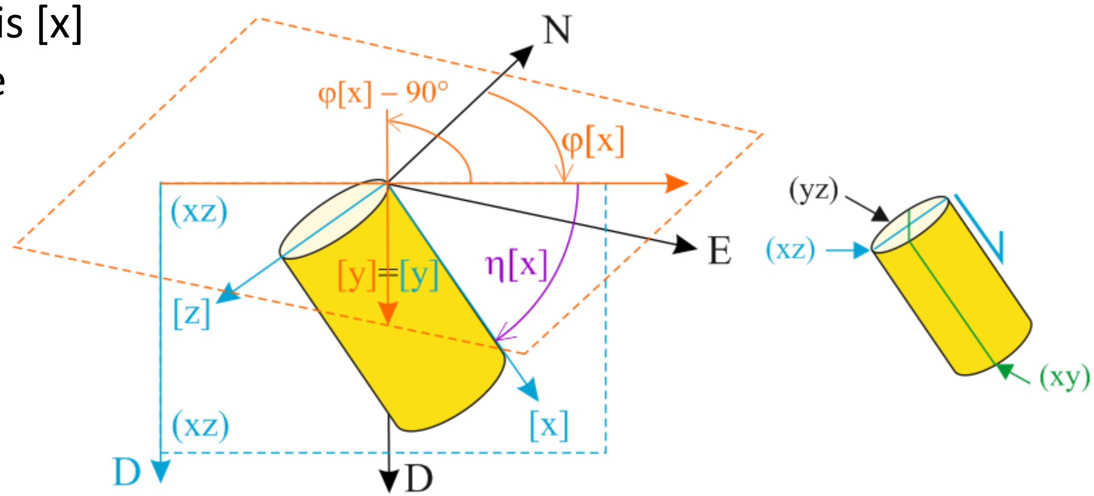
$$\begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi \\ \sin\theta\sin\psi \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X$$

$$\begin{bmatrix} -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi \\ -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi \\ \sin\theta\cos\psi \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Y$$

$$\begin{bmatrix} \sin\phi\sin\theta \\ -\cos\phi\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad Z$$

Each column of the rotation matrix contain the orientation of each axis

Case a) Core with the orientation arrow drawn on its elongation axis [x] pointing down towards the trend direction $\eta[x] > 0$



Trend plunge conversion in strike dip rake

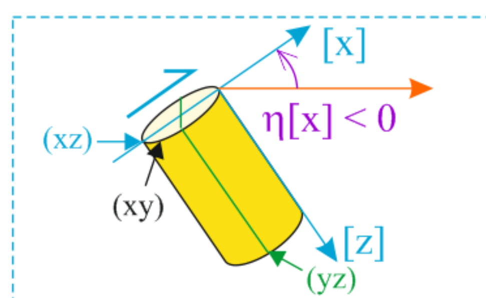
$\phi = \varphi - 90°$ or $\varphi + 270°$

$\theta = \eta$

$\psi = +90°$

The inclination of a plan towards the trend direction required a 1st rotation around the [Y] axis and thus a strike of a plane on the left (–90°) of the trend or plunge direction with a dip angle equal to the plunge angle. [X] being horizontal a 2nd rotation on the plane (XY) is required to turn back [X] in the dip direction and [Y] on its right in the horizontal

Case b) Core with the orientation arrow [x] plot on its top and pointing up towards the trend direction $\eta[x] < 0$



In structural geology X, Y and Z are the main axes of a deformation along which a, b and c are the lengths of the deformation ellipsoid.

In image analysis X, Y and Z are the coordinate axes of pixels x, lines y and stacks of planes z.

It was therefore necessary in image analysis of SPO to define a convention avoiding confusions between axis:

[A], [B] and [C] are the main axes along which a, b and c are the lengths of the deformation ellipsoid.

[X], [Y] and [Z] are the coordinate system of an image along which we may count pixels x, lines y and stacks of planes z.

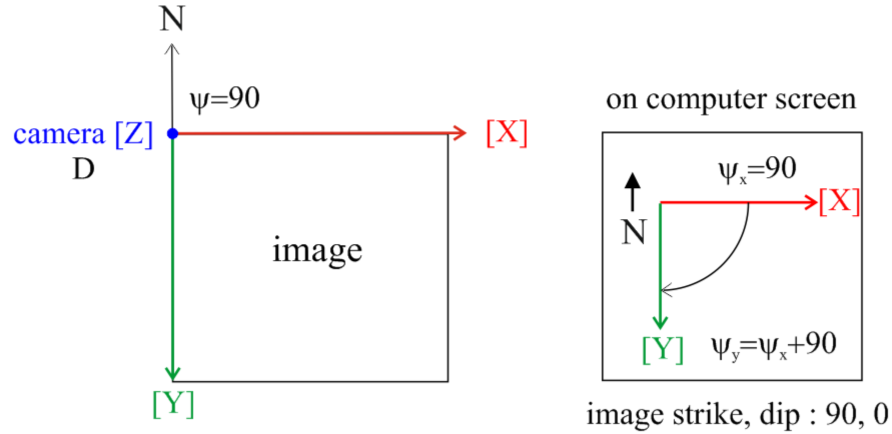
[N], [E] and [D] are the north east down of the geographic coordinate system.

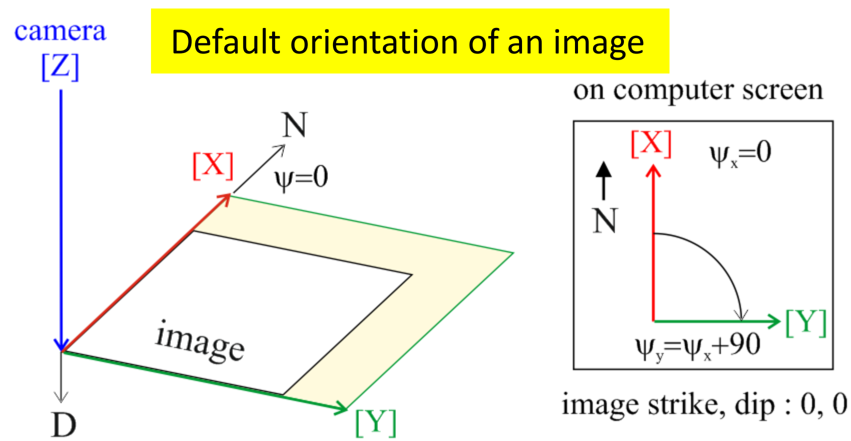
(A,B,C), (X,Y,Z) and (N,E,D) can be oblique to one another

Image orientation

The lens of the camera should be looking down in the direction [Z] towards the image plane (X,Y) where X are columns of pixels and Y stacks of lines. In this case the camera is vertical and the image is horizontal with pixels aligned on the east when ψ_x=90° and to the north when ψ_x=0°.
[Y] is always at +90° of [X].

When the lens of the camera is looking up from the bottom of the image, [Z] is also pointing upwards causing an upside-down image with an inversion of the rotation (-90°) from [X] to [Y].
This is one of the key issue preserving the consistency of the data in 3D.





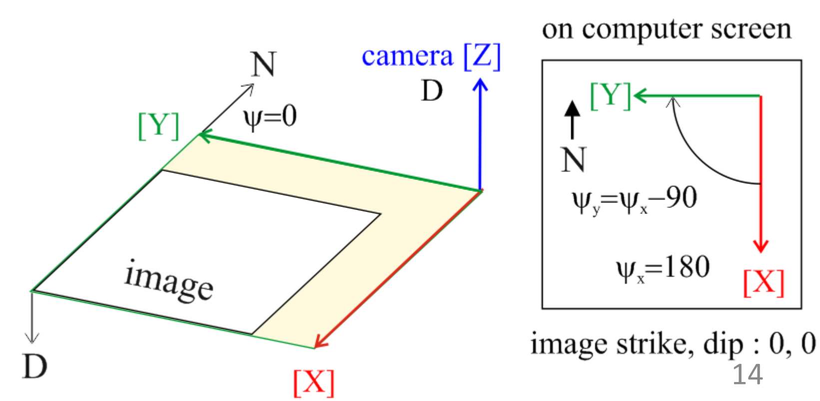
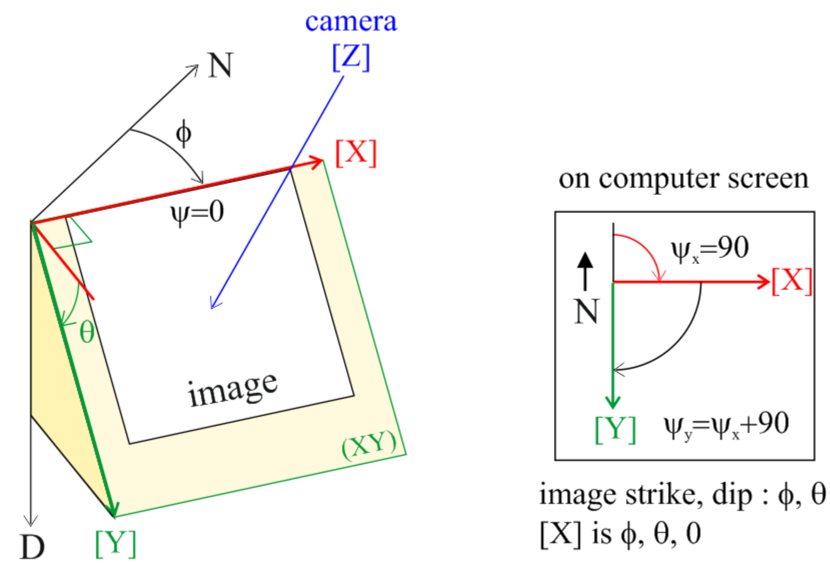


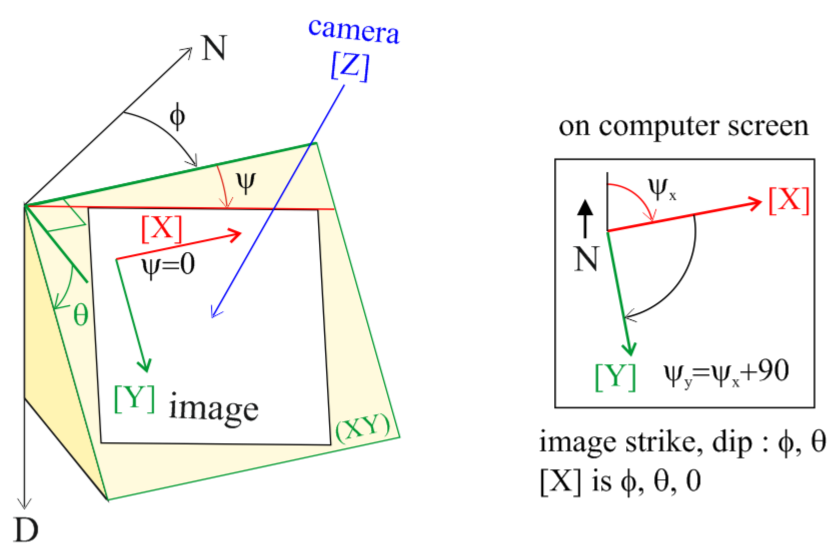
Image orientation

The lens of the camera should always be perpendicular to the image plane.

When the top of the image is horizontal and its plane is dipping towards the operator ψ is equal to 0, the strike is on the right and the dip is down. On a map and its display on a computer screen, the N is by default upwards. A rake (ψ_x) of 90° allows to preserve the horizontality of the image on the screen.

When the top is not perfectly horizontal the angle ψ must be measured for a correct orientation of the data. A lake of careful orientation of the image can conduct to false results.



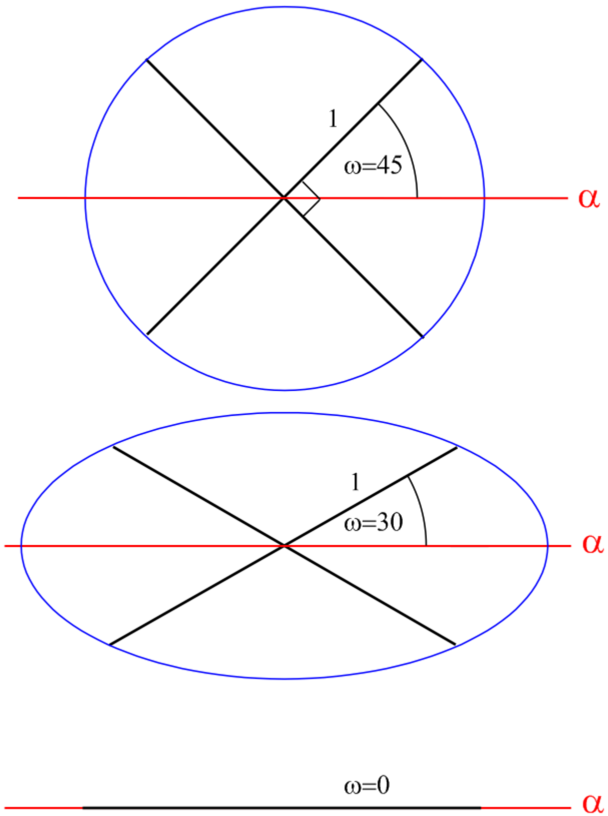


Consider the orientation of 2 lines representing the smallest population of 2 objects (crystals, grains, …).

2 perpendicular lines represent an isotropic angular distribution forming a circle. Each line makes an angle of ω degrees with the mean orientation of the population.

When both lines turn towards each other they define a preferred orientation α with an anisotropic angular distribution forming an ellipse.

When both lines are parallel to each other they define an infinite preferred orientation with ω equal to 0.



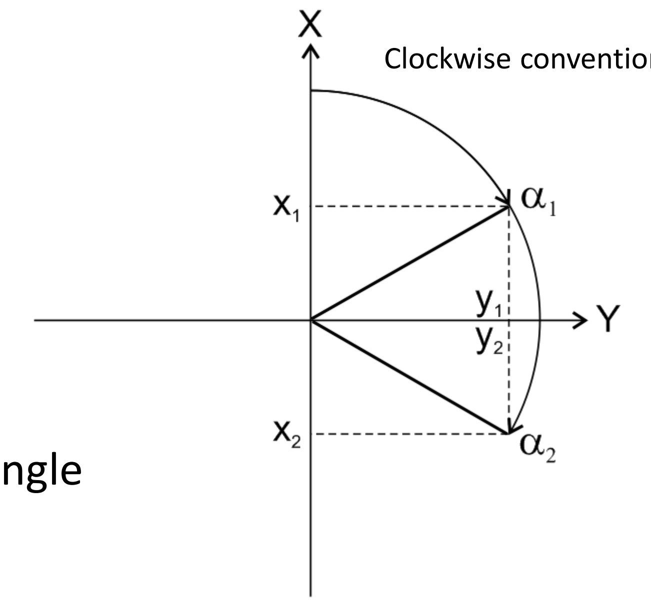
Considering the cosine directions of the lines Harvey and Laxton (1980) defined a cosine direction distribution matrix:

$$\mathbf{M} = \begin{vmatrix} m_{xx} & m_{xy} \\ m_{xy} & m_{yy} \end{vmatrix}$$

$$m_{xx} = \frac{1}{J}\sum_j (\cos\alpha_j)^2$$

$$m_{xy} = \frac{1}{J}\sum_i (\cos\alpha_i)(\sin\alpha_i)$$

$$m_{yy} = \frac{1}{J}\sum_i (\sin\alpha_i)^2$$



Its first **eigenvector** gives the preferred orientation (PO) angle α and the **eigenvalue** (μ) ratio measures the PO intensity

$$\begin{vmatrix} \mu_a & 0 \\ 0 & \mu_b \end{vmatrix} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} \begin{vmatrix} m_{xx} & m_{xy} \\ m_{xy} & m_{yy} \end{vmatrix} \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

$$\Lambda = R^{-1}\, M\, R$$

$$\begin{vmatrix} \mu_a & 0 \\ 0 & \mu_b \end{vmatrix} = \begin{vmatrix} m_{xx}\cos^2\alpha + m_{yy}\sin^2\alpha + m_{xy}\sin 2\alpha & \frac{1}{2}(m_{yy}-m_{xx})\sin 2\alpha + m_{xy}\cos 2\alpha \\ \frac{1}{2}(m_{yy}-m_{xx})\sin 2\alpha + m_{xy}\cos 2\alpha & m_{xx}\sin^2\alpha + m_{yy}\cos^2\alpha - m_{xy}\sin 2\alpha \end{vmatrix}$$

→ $\frac{1}{2}(m_{yy}-m_{xx})\sin 2\alpha + m_{xy}\cos 2\alpha = 0$ → $\tan 2\alpha = \frac{2.m_{xy}}{m_{xx}-m_{yy}}$ → $\alpha = \frac{1}{2}\arctan\left(\frac{2.m_{xy}}{m_{xx}-m_{yy}}\right)$

knowing α → $\mu_a = m_{xx}\cos^2\alpha + m_{yy}\sin^2\alpha + m_{xy}\sin 2\alpha$, $\mu_b = m_{xx}\sin^2\alpha + m_{yy}\cos^2\alpha - m_{xy}\sin 2\alpha$ → $Rf = \frac{\mu_a}{\mu_b}$ → $\mu_a = \sqrt{Rf}$, $\mu_b = \frac{1}{\sqrt{Rf}}$

The cosine directions PO can be summarized as:

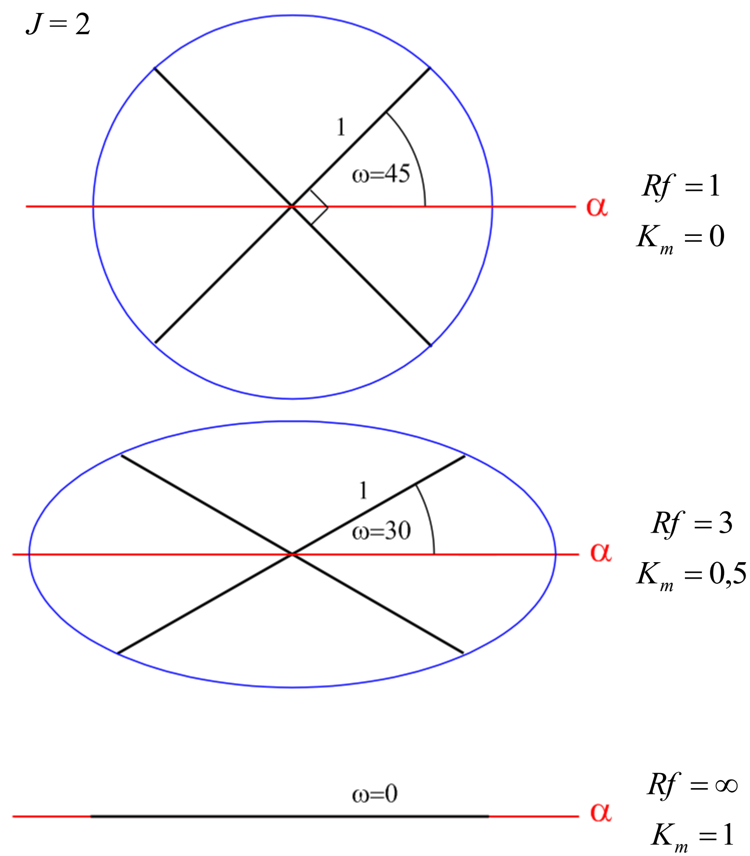
$$\mathbf{M} = \frac{1}{J}\begin{bmatrix} \sum\cos^2\alpha_j & \sum\cos\alpha_j\sin\alpha_j \\ \sum\sin\alpha_j\cos\alpha_j & \sum\sin^2\alpha_j \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{Rf} & 0 \\ 0 & 1/\sqrt{Rf} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} m_{xx} & m_{xy} \\ m_{xy} & m_{yy} \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Harvey & Laxton (1980)

ω is a remarkable angle of the radius remaining constant from a circle to an ellipse during a deformation. It is linked to the ellipse shape ratio as it follows:

$$Rf = \frac{\cos^2\omega}{\sin^2\omega} = \frac{1}{\tan^2\omega}$$



Rf varying from 1 to the infinity, many calculation rather use instead the $\mathbf{M}$ shape parameter K_m which varies from 0 to 1.

$$K_m = \frac{Rf - 1}{Rf + 1}$$

Willis (1977) Fernandez *et al.* (1983)

Cosine directions PO in 3D in the X,Y,Z coordinate system

The method of Jacobi determines the rotation of each plane *xy*, *xz* and *yz* successively until it converges towards a 0 rotation.

$$\mathbf{M} = \begin{vmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{xy} & m_{yy} & m_{yz} \\ m_{xz} & m_{yz} & m_{zz} \end{vmatrix}$$

$$\mathbf{R}_{xy} = \begin{vmatrix} \cos(\alpha_{xy}) & -\sin(\alpha_{xy}) & 0 \\ \sin(\alpha_{xy}) & \cos(\alpha_{xy}) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

The determination of the first rotation on *xy* is a simple 2D PO calculation (see p.17)

$$\begin{vmatrix} \mu_{a_{xy}} & 0 \\ 0 & \mu_{b\,xy} \end{vmatrix} = \begin{vmatrix} \cos\alpha_{xy} & \sin\alpha_{xy} \\ -\sin\alpha_{xy} & \cos\alpha_{xy} \end{vmatrix} \begin{vmatrix} m_{xx} & m_{xy} \\ m_{xy} & m_{yy} \end{vmatrix} \begin{vmatrix} \cos\alpha_{xy} & -\sin\alpha_{xy} \\ \sin\alpha_{xy} & \cos\alpha_{xy} \end{vmatrix}$$

$$\mathbf{R}_{xz} = \begin{vmatrix} \cos(\alpha_{xz}) & 0 & -\sin(\alpha_{xz}) \\ 0 & 1 & 0 \\ \sin(\alpha_{xz}) & 0 & \cos(\alpha_{xz}) \end{vmatrix}$$

$$\alpha_{xy} = \frac{1}{2}\arctan\left(\frac{2.m_{xy}}{m_{xx} - m_{yy}}\right)$$

$$\mathbf{R}_{yz} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_{yz}) & -\sin(\alpha_{yz}) \\ 0 & \sin(\alpha_{yz}) & \cos(\alpha_{yz}) \end{vmatrix}$$

Application of the rotation α_{xy} to **M** at 3 dimensions $\mathbf{M}_{i+1} = \mathbf{R}_{xy}^{-1} \cdot \mathbf{M}_i \cdot \mathbf{R}_{xy}$

Repeat of the same operation on planes *xz* and *yz*

Continuation of the loop until the rotations become zero $\mathbf{R}_i = \mathbf{R}_{i-1} \cdot \mathbf{R}_{xy} \cdot \mathbf{R}_{xz} \cdot \mathbf{R}_{yz}$

The cumulative rotations gives the eigenvectors and orientation of the final diagonalized **M** giving the eigenvalues

with $\mathbf{R}_0 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

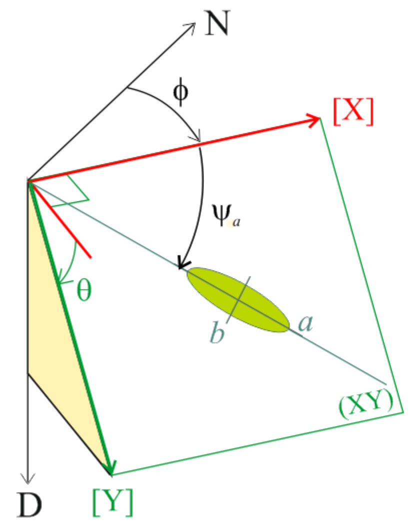
$$\mathbf{M}_{final} = \begin{vmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{vmatrix}$$

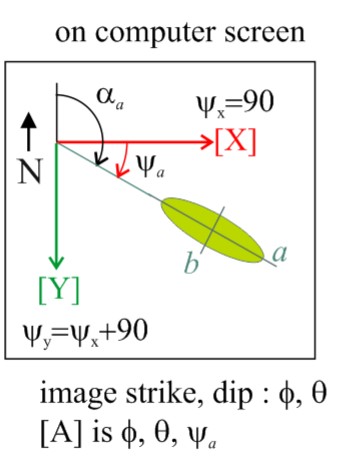
The cosine directions PO has an intensity and an orientation which can be represented by an ellipse on the section of measurement.

The anisotropy (ellipse a,b) found in each section (X,Y) is often oblique like in this example.

The orientation of the PO long axis [A] is given by α within the image and must be converted in ψ_a to be oriented in the geographic system.

$$\psi_a = \alpha_a - \psi_x$$
$$\psi_b = \psi_a + 90$$





The length of the long a and short b axes of the PO ellipse are given by

$$a = \sqrt{Rf}$$
$$b = \frac{1}{\sqrt{Rf}}$$

Harvey & Laxton (1980)

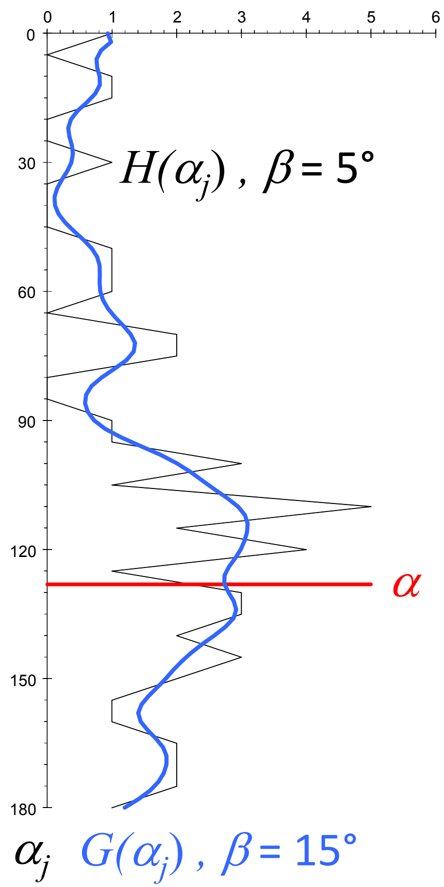
2D PO preferred orientation

Histogram of 50 directions $H(\alpha_i)$ counted with a class size of 5°

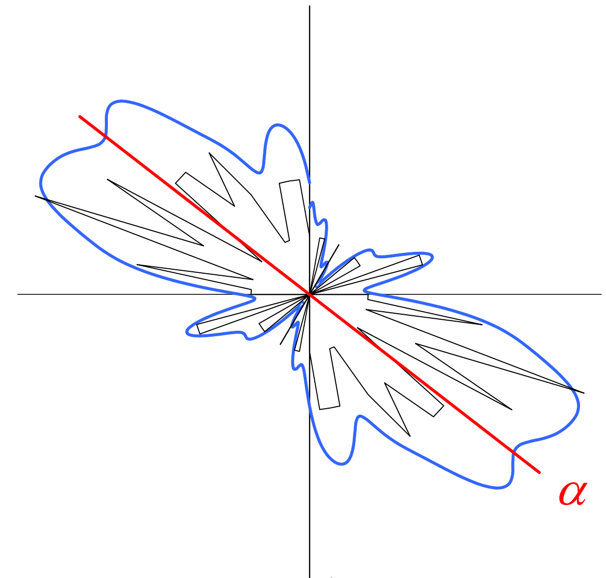
Robin and Jowett (1985) β angular Gaussian filtering of the direction histogram plot with a resolution of 1°

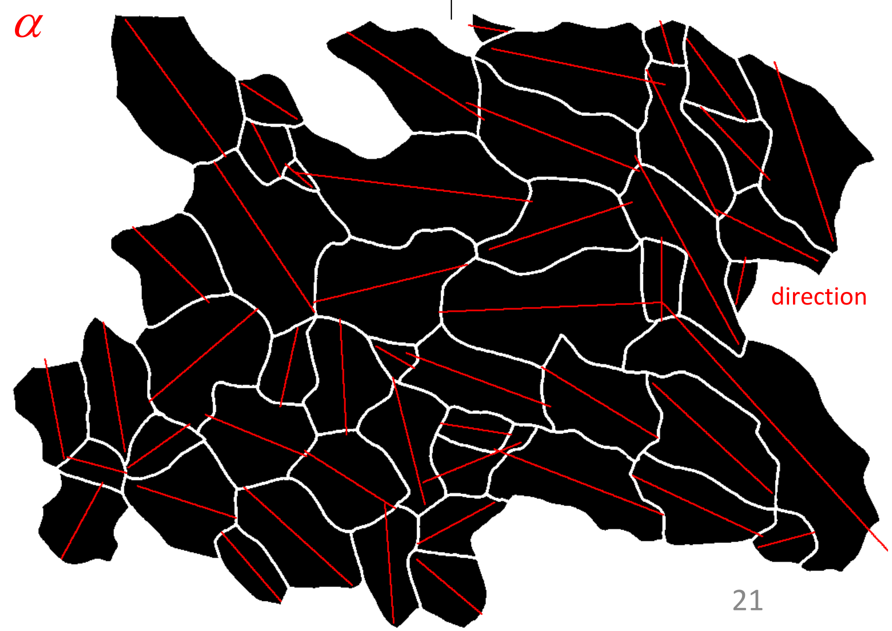
$$G(\alpha_i) = \sum_{j=0}^{359} \exp\left(k \cdot \left(\cos\left(2 \cdot (\alpha_j - \alpha_i)\right) - 1\right)\right)$$

with $$k = \frac{1}{1 - \cos(\beta/2)}$$



rose of directions





$Rf = 2.421$, $Rf^{1/2} = 1.556$, $\alpha = 143°$

Harvey & Laxton (1980)

3D PO preferred orientation of directions without length

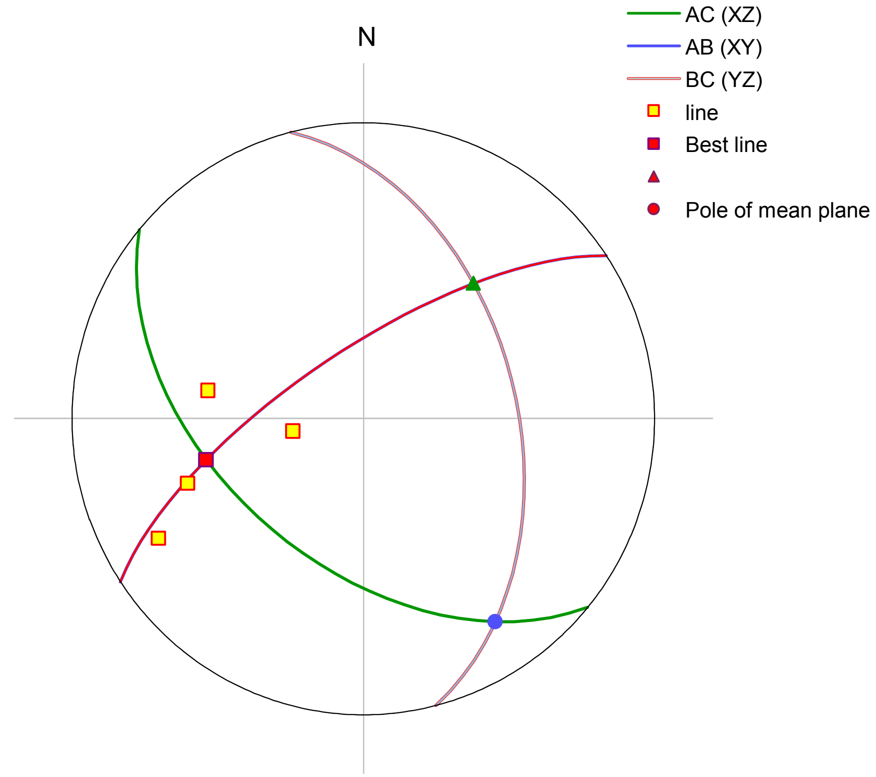
The 3D matrix of cosine direction distribution $\mathbf{M}_{NED}$ can quantify the preferred orientation by the calculation of 3 eigenvectors and 3 eigenvalues with the method of Jacobi presented in p 18.

$$\mathbf{M}_{NED} = \frac{1}{J} \cdot \begin{vmatrix} \sum n_j^2 & \sum n_j \cdot e_j & \sum n_j \cdot d_j \\ \sum n_j \cdot e_j & \sum e_j^2 & \sum e_j \cdot d_j \\ \sum n_j \cdot d_j & \sum e_j \cdot d_j & \sum d_j^2 \end{vmatrix}$$

with

$$n_j = \cos\eta_j \cdot \cos\varphi_j$$
$$e_j = \cos\eta_j \cdot \sin\varphi_j$$
$$d_j = \sin\eta_j$$

Cosines directions



Example	right hand rule
80 SW 70	260 / 70
60 SW 20	240 / 20
100 W 45	280 / 45
70 W 36	250 / 36

	A	B	C
μ eigen	0.871	0.107	0.022
φ trend	240.2	338.8	140.3
η plunge	9.4	42.3	46.2
R_{ac}	39.7		
R_{ab}	8.1		
R_{bc}	4.9		

Harvey & Laxton (1980)