Constraint Programming:

Local Consistencies

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Objective

Solving constraint over finite domains

- exhaustive search vs. filtering algorithms
- recap about constraint propagation
- incomplete solvers and local consistency notion
- node consistency (NC algorithm)
- arc consistency (algorithms: AC-1, AC-3, AC-4)
- bound consistency

Finite domains

each set isomorphic to a finite part of $\mathbb{N}$:

1. Set of natural integer that can be represented by a machine
2. Booleans : $\{\text{false}, \text{true}\}$ (or $\{0, 1\}$)
3. Letters : $A, B, C, \ldots$
4. Set of the members of a team
5. ...

$\Rightarrow$ FD = very important to model numerous industrial problems

CSP (reminder)

A *constraint satisfaction problem* (CSP) is defined by:

- a sequence of variables $X = x_1, \ldots, x_n$ with *domains* $D_1, \ldots, D_n$ (associated to the variables)
- a set of constraints $C_1, \ldots, C_l$, each C_i on a sub-sequence Y_i of X

implicitely, the CSP represents the constraint:

$$C_1 \wedge \ldots \wedge C_n \wedge x_1 \in D_1 \wedge \cdots \wedge x_n \in D_n$$

A *solution of the CSP* is a n-tuple $d = (a_1, \ldots, a_n)$ such that:

- $d \in D_1 \times \cdots \times D_n$
- and for each i, $d[Y_i] \in C_i$
  ($d[Y_i]$ satisfies C, or $C(a_{i_1}, \ldots, a_{i_l})$ is true)

Solving CSPs (1)

Look back: variables are instanciated, and "instanciated" constraints are tested

- non-incremental version: *generate and test*
- incremental version: *backtracking*

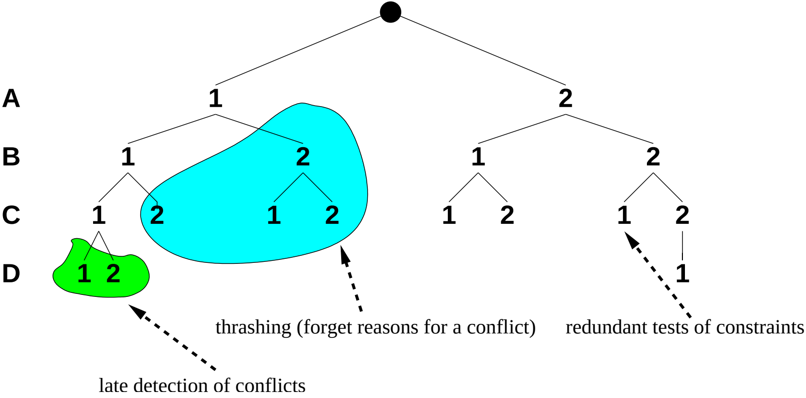
$\Uparrow$ complete and correct

$\Downarrow$ inefficient and costly

clever alternatives: *backjumping*, *backmarking*

Problem of the *look back*

$A \in \{1, 2\}, B \in \{1, 2\}, C \in \{1, 2\}, D \in \{1, 2\}$
$A > D \wedge B = C \wedge A = C$



Solving CSPs (2)

basic idea: from a given CSP, find an *equivalent* CSP with smaller domains (smaller search space)

- consider each atomic constraint separately
- filter domains of variables and eliminate *inconsistent* values

$\Uparrow$ active use of the constraints. Many values vialoting constraints are removed

$\Downarrow$ incomplete (complete with split and search)

Constraint solving framework

```
solve(CSP):
        while not finished do
            pre-process
            constraint propagation
            if happy
               then finished=true
               else split
                    part-of search
            endif
        endwhile
```

where part-of search consists in calls to the solve function

Remark: part-of search is one of the mechanisms defining search

Constraint propagation

- replace a CSP by a CSP which is:
  - equivalent (same set of solutions)
  - “smaller” (domains are reduced)
  - “simpler” (constraints are reduced)
- constraint propagation mechanism:
  repeatedly reduce domains or constraints
- can be seen as a fixed point of application of reduction functions
  - reduction function to reduce domains or constraints
  - can be seen as an abstraction of the constraints by reduction functions

Constraint propagation: reducing domains

- Generally:
  - reduce domains using constraint and domains
  - $\rightarrow$ reduce the search space
- generic domain reduction:
  - Given a constraint C over $x_1, \ldots, x_n$ with domains $D_1, \ldots, D_n$
  - select a variable x_i to be reduced
  - delete from D_i all values for x_i that do not participate in a solution of C

Local consistency

- a criterion to stop propagation
- a way to characterize a CSP or a constraint
- why local?
  - generally, unable to obtain global consistency (incomplete solvers without split and search)
  - thus, local means on a sub-set of a CSP
    $\rightarrow$ usually, local to ONE constraint
    this sub-set is used to reduce domains

Local consistency (1)

at the beginning: for unary and binary constraints

- unary constraints: *node consistency*
  - for constraints such as: even(x), $y > 5$, ...
- binary constraints: *arc consistency*
  - for constraints such as: $x > y + 4$, $x \neq y$, ...

Local consistency (2)

then: for *n*-ary constraints and higher/stronger consistencies

- *n*-ary constraints: *hyper-arc consistency*
  - for constraints such as: $3.x + y = z$, $and(x, y, z)$, ...
- (m-)path consistency:
  - using several constraints at a time
- k-consistency:
  - every $(k-1)$-consistent instanciation can be extended to a k-consistent instanciation (k variables)

Local consistency (3)

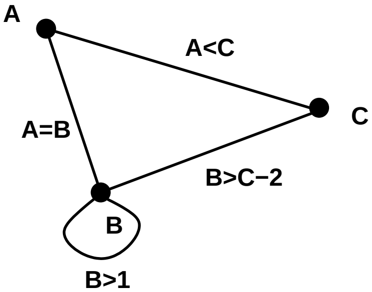
then: consistency on bounds of domains
(when domains are too big to consider each value)

- bound consistency (finite domains):
  - for constraints such as: $3.x + y = z$ with domains $x \in [-10000..9000], y \in [-5000..9000], z \in [100..19000], \ldots$
- 2b consistency (real interval, "primitive" constraints)
  - for constraints such as: $3.23 * x * y = z$ with domains $x \in [-100.1547..9000.0], y \in [-5.12..9.0], z \in [0.99..1.01]$
- box consistency (real interval)
  - for constraints such as: $3.23 * x + y * x = z^2 + exp(x)$ with domains $x \in [-10.147..90.0], y \in [-5.1..9.0], z \in [0.99..1.01]$

Local consistency: intuitive (1)

$\{B > 1, A < C, A = B, B > C - 2;\ A, B, C \in \{1, 2, 3\}\}$

the CSP can be represented by the graph:

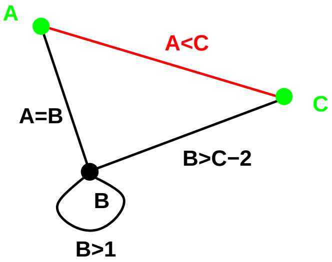


how to reduce domains?

Idea: follow arcs of the graph

Local consistency: intuitive (2)

using $A < C$, A and/or C may be reduced



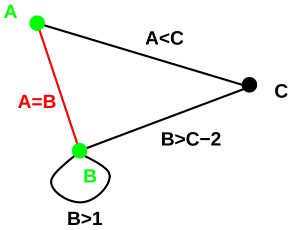
A and C reduced to $A \in \{1, 2\}$, $C \in \{2, 3\}$

now, reduce B using $A = B$ or $B > C - 2$

Local consistency: intuitive (3)

$\{B > 1, A < C, A = B, B > C - 2;\ A \in \{1, 2\}, B \in \{1, 2, 3\}, C \in \{2, 3\}\}$

using $A = B$, A and/or B may be reduced



B reduced to $B \in \{1, 2\}$

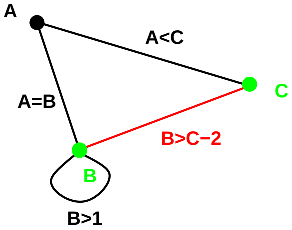
A not reduced, so useless to use $A < C$

now, reduce C and/or B using $B > C - 2$, or reduce B using $B > 1$

Local consistency: intuitive (4)

$\{B > 1, A < C, A = B, B > C - 2;\ A \in \{1,2\}, B \in \{1,2\}, C \in \{2,3\}\}$

using $B > C - 2$, B and/or C may be reduced



B, C not reduced

$B > 1$ can be used,
and so on until no domain can be reduced anymore

Node consistency: definition

Definition : an atomic unary constraint C over the variable x with the domain D_x is node consistent iff:

$$\forall a \in D_x : a \in C \quad (\text{or } C(a))$$

Remarks:

- a non unary constraint is always considered as node consistent
- a CSP is node consistent if all its constraints are node consistent

Examples:

- $x \in \{4, 6\}$, even(x) is node consistent
- $x \in [2..12], x > 5$ is not node consistent

Node consistency: algorithm

```
node_consistency(C, D)
begin
    let C ≡ C_1, ··· , C_n
    for i ← 1 to n do
        D ← revise_node(C_i, D)
    endfor
    return(D)
end
```

```
revise_node(C, D)
begin
    if (|var(C)| == 1) then
        {x} ← var(C)
        D_x ← {d ∈ D_x | d ∈ C}
    endif
    return(D)
end
```

Arc Consistency: definition

Definition : an atomic binary constraint C over the variables x and y with domains D_x and D_y is arc consistent iff:

- $\forall a \in D_x \exists b \in D_y$ s.t. $(a, b) \in C$
- $\forall b \in D_y \exists a \in D_x$ s.t. $(a, b) \in C$

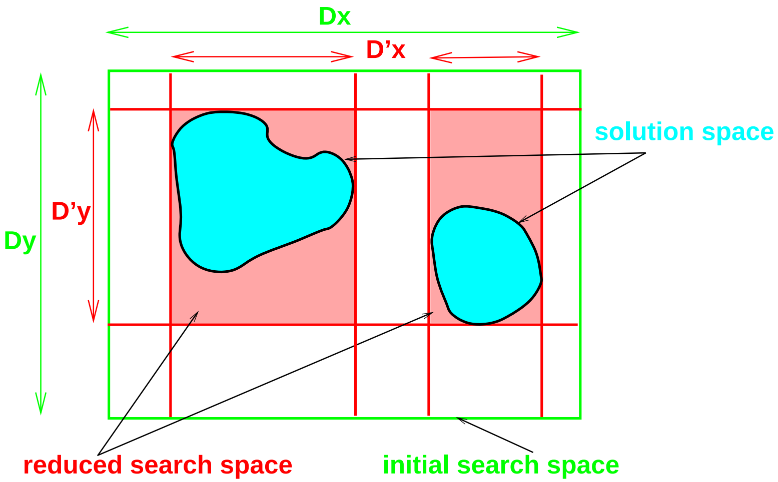
Remarks:

- a non binary constraint is arc consistent
- a CSP is arc consistent iff all its constraints are arc consistent

Examples:

- $x \in \{1, 3\}, y \in \{2, 4\}, x + y = 5$ is arc consistent
- $x \in \{1, 2\}, y \in \{1, 7\}, x = y$ is not arc consistent

Arc Consistency: intuition



Arc consistency: AC-1 algorithm

```
AC-1(C, D)
begin
    let C ≡ C_1, ··· , C_n
    repeat
        D′ ← D
        for i ← 1 to n do
            D ← revise_arc(C_i, D)
        endfor
    until(D′ = D)
    return(D)
end
```

```
revise_arc(C, D)
begin
    if (|var(C)| == 2) then
        {x, y} ← var(C)
        D_x ← {a ∈ D_x |
                    ∃b ∈ D_y : (a, b) ∈ C}
        D_y ← {b ∈ D_y |
                    ∃a ∈ D_x : (a, b) ∈ C}
    endif
    return(D)
end
```

Local consistency: example

Consider the CSP

$$\{X < Y, Y < Z, Z \leqslant 2;\ D_X, D_Y, D_Z \in \{1, 2, 3\}\}$$

Computation of node consistency

$Rightarrow$ 3 removed from D_z

Computation for arc consistency

$\Rightarrow$ inconsistent

Generally: incompleness. Algorithm returns some domains for the variables. All kept values are not necessarily solution!

Arc consistency $\neq$ consistency

Consider the CSP

$$\{x = y, x \neq y, D_x \in \{a, b\}, D_y \in \{a, b\}\}$$

the CSP is arc consistency

$\Rightarrow$ a and b cannot be reduced using $x = y$ or $x \neq y$

However, the CSP is not consistent

$\Rightarrow$ no solution

Problems of AC-1

- inefficient
- wake-up constraints when useless
  - no modification of variable domains
- no early detectection of failed CSP
  - two loops with failed CSP

Idea of AC-3

Idea: wake up constraints when variables have effectively been modified

Mechanism:

- manage a set of constraints to use
- update this set after each reduction attemp
  - add constraints with at least one modified variable
- stop
  - when no more constraint to consider
- failed CSP
  - stop as soon as one domain is empty

Local consistency: AC-3 algorithm

AC-3($C \equiv C_1, \cdots, C_n, \mathbf{D}$)
begin
  $\mathcal{S} \leftarrow \{C_1, \cdots, C_n\}$
  while ($\mathcal{S} \neq \varnothing$)
    choose and extract C from $\mathcal{S}$
    $\mathbf{D}' \leftarrow \text{revise\_arc}(C, \mathbf{D})$
    if ($\mathbf{D}' = \varnothing$) **then return**($\varnothing$) **endif**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{C_i \mid \exists x \in \text{var}(C_i) \text{ s.t. } \mathbf{D}'_x \neq \mathbf{D}_x\}$
    $\mathbf{D} \leftarrow \mathbf{D}'$
  endwhile
  return($\mathbf{D}$)
end

Revise_arc unchanged

Local consistency: AC-4 algorithm

Possible speed-up for AC-3: to keep in memory for each binary constraints $c(x, y)$ *support* relations between values of D_x and D_y :

- how many values of D_y support each value of D_x
- what are the values of D_x supported by a particular value of D_y

and vice-versa.

$\Uparrow$ when a value is removed, we know precisely the changes that are induced, and which constraints to wake-up

$\Downarrow$ memory space
AC-4 : best theoretical complexity... often the worst in pratice

Hyper-arc consistency (1)

what about n-ary constraints for n ¿ 2 ?

hyper-arc consistency: a constraint C over the variables $x_1, \ldots, x_n$ with domains $D_1, \ldots, D_n$ is *hyper-arc consistent* w.r.t. x_i ($i \in \{1, \ldots, n\}$) iff:

$$\forall a \in D_i, \quad \exists d \in D_1 \times \ldots \times D_n \quad \text{s.t.} \quad d \in C \quad \text{and} \quad a = d[x_i]$$

Hyper-arc consistency (2)

- a constraint C over $x_1, \ldots, x_n$ with domains $D_1, \ldots, D_n$ is hyper-arc consistent iff c is hyper-arc consistent w.r.t. x_i for all $i \in \{1, \ldots, n\}$.
- a CSP is hyper-arc consistent iff all its constraints are hyper-arc consistent

Hyper-arc consistency (3)

Examples: constraints

- $x \in \{3, 5, 7\}, y \in \{1, 4\}, z \in \{4, 6, 14\}$
  $x + 2 * y = z + 1$ is hyper-arc consistent
- $x \in \{1, 2, 4\}, y \in \{3, 5\}, z \in \{4, 5\}$
  $x + y - z = 0$ is not hyper-arc consistent
  (not hyper-arc consistent w.r.t. x, e.g., value 4)

Examples: CSP

- { and(x,y,z), or(x,y,1); $x \in \{1\}, y \in \{0, 1\}, z \in \{0, 1\}$}
  the CSP is hyper-arc consistent
- { and(x,y,z), or(x,y,1); $x \in \{0, 1\}, y \in \{0, 1\}, z \in \{1\}$}
  the CSP is not hyper-arc consistent

Directional arc consistency (1)

Idea: directional propagation

consider an ordering $<$ on variables:

directional arc consistency: a constraint C over the variables x, y with domains D_x, D_y is directionally arc consistent w.r.t. $<$ iff:

- if $x < y$:
$$\forall a \in D_x \exists b \in D_y \ (a, b) \in C$$
- if $y < x$:
$$\forall b \in D_y \exists a \in D_x \ (a, b) \in C$$

a CSP is directionally arc consistent w.r.t. $<$ iff all its constraints are

Directional arc consistency (2)

example:

$$\{x < y; \quad x \in [2..7], y \in [3..7]\}$$

- the CSP is not arc consistent
- the CSP is directionally arc consistent w.r.t. $y < x$
- the CSP is not directionally arc consistent w.r.t. $x < y$

Limitations of arc/hyper-arc consistency (1)

Problem: determining arc/hyper-arc consistency can be too costly

Example:
$\{x = y + z, 2.x = 4.y;\ x, y, z \in \{1, 2, 8, 12, 34, \ldots, 110000\}\}$
domain reduction: each value must be tested!!!

Idea: to relax consistency $\Rightarrow$ test only *bounds*

Limitations of arc/hyper-arc consistency (2)

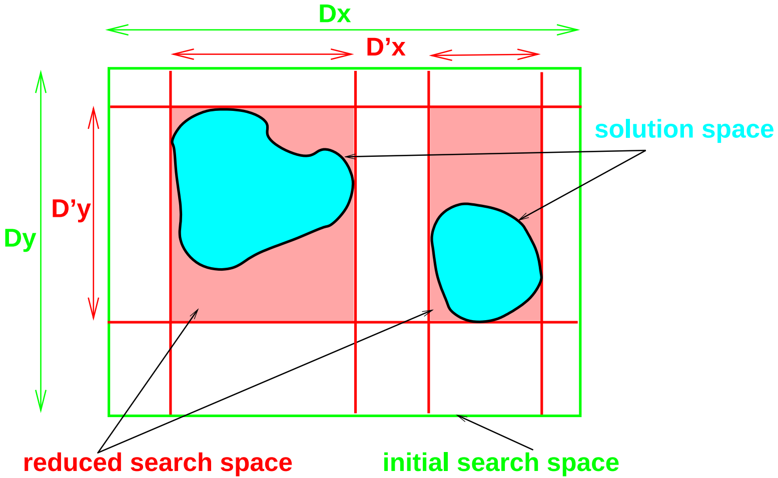
Example: $\{x < y, y < z, z < x;\ x, y, z \in [1..10000]\}$

domain reduction:

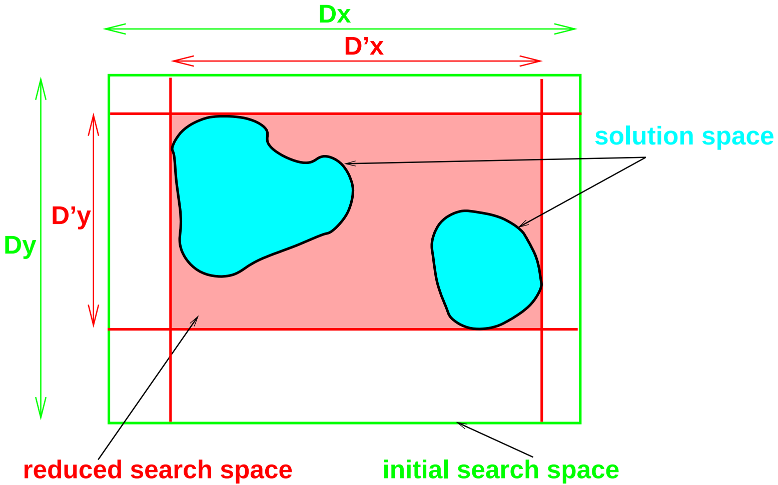
- using the first constraint:
  $\{x < y, y < z, z < x;\ x \in [1..9999], y \in [2..10000], z \in [1..10000]\}$
- using the second constraint:
  $\{x < y, y < z, z < x;\ x \in [1..9999], y \in [2..9999], z \in [3..10000]\}$
- using the third constraint:
  $\{x < y, y < z, z < x;\ x \in [4..9999], y \in [2..9999], z \in [3..9998]\}$
- ... until a domain is empty

Idea 1: testing bounds does not change the cost
Idea 2: symbolic computation $\rightarrow$ direct proof (transitivity of $<$)
Idea 3: using two constraints at a time $\rightarrow$ *path consistency*

Arc consistency: intuition (recap)



Bound consistency: intuition



Bound consistency (1)

Idea : domains are represented by intervals

bound consistency: a constraint C over the variables $x_1, \ldots, x_n$ with domains $D_1, \ldots, D_n$ is bound consistent w.r.t. x_i with domain $D_i = [l, r]$ ($i \in \{1, \ldots, n\}$) iff:

$$\exists d \in D_1 \times \ldots \times D_n \quad \text{s.t.} \quad d[x_i] = l \;\text{ and }\; d \in C$$

and

$$\exists d \in D_1 \times \ldots \times D_n \quad \text{s.t.} \quad d[x_i] = r \;\text{ and }\; d \in C$$

Bound consistency (2)

- a constraint c is bound consistent iff it is w.r.t. x_i for all $i \in \{1, \ldots, n\}$.
- a CSP is bound consistent iff all its constraints are bound consistent

Examples :

- $x \in [3..6], y \in [2, 3], z \in [5, 9], x + y = z$
  is bound consistent
- $x \in [2..3], y \in [3..6], z \in [1..19], 3 * x = y + z$
  is not bound consistent
  (not bound consistent w.r.t. z, e.g., value 19)

Bound consistency (3)

computing bound consistency for “primitive” constraints: reasonning only on bounds
$\Rightarrow$ easy, less complexe

Examples:

- $x + y = z$ with $D_x = [a..b]$, $D_y = [c..d]$, $D_z = [e, f]$
- $x \leqslant y$ with $D_x = [a..b]$, $D_y = [c..d]$

Bound consistency: $x \leqslant y$

constraint:

$$x \leqslant y$$

to get bound consistency:

$$x \leqslant \max D_y$$
$$y \geqslant \min D_x$$

Bound consistency: $x \leqslant y$

$\% \; x \leqslant y$

revise_leq($D_x = [a..b], D_y = [c..d]$)

begin

$D_x \leftarrow [a..\min\{b, d\}]$

$D_y \leftarrow [\max\{a, c\}..d]$

return(D_x, D_y)

end

Bound consistency: $x + y = z$

constraint:

$$x + y = z \quad \equiv \quad x = z - y \quad \equiv \quad y = z - x$$

to get bound consistency:

$$\begin{aligned} z &\geqslant \min D_x + \min D_y & \qquad z &\leqslant \max D_x + \max D_y \\ x &\geqslant \min D_z - \max D_y & \qquad x &\leqslant \max D_z - \min D_y \\ y &\geqslant \min D_z - \max D_x & \qquad y &\leqslant \max D_z - \min D_x \end{aligned}$$

Bound consistency: $x + y = z$

% x+y=z

revise_addition($D_x = [a..b], D_y = [c..d], D_z = [e..f]$)
begin
  $D_x \leftarrow D_x \cap [e - d..c - f]$
  $D_y \leftarrow D_y \cap [e - b..f - a]$
  $D_z \leftarrow D_z \cap [a + c..b + d]$
  return(D_x, D_y, D_z)
end

Combination BT/AC

solvers using only local consistency: incomplete
realizing a complete solver $\Rightarrow$ combination with backtracking

Look ahead: instanciation of some variables with filtering of domains
$\Rightarrow$ *forward checking*, *partial look-ahead*, *full look-ahead*

$\Uparrow$ no exploration of branches trivialy without solution

$\Downarrow$ more work after each instanciation