# Mechanics of Composites - Lecture notes

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## 1 Effective behavior of unidirectional composites

### 1.1 Longitudinal behavior

It is commonly admitted and verified that the simple following mixture rule provides a good estimation in the longitudinal direction. This model is based on the perfect matching of deformations of both the matrix and the fibers along the fibers direction, when the material is loaded along the fibers direction. One can easily check that:

$$E_c^L = V_f E_f + (1 - V_f) E_m \tag{1}$$

### 1.2 Transverse behavior

When the UD is mechanically loaded along the transverse direction, the situation is more complex and oversimplifying assumptions such as the *serial model* (parallel layers of fibres and matrix) give poor results:

$$E_c^T = \frac{E_f E_m}{V_f E_m + (1 - V_f) E_f} \tag{2}$$

As shown on figure 1.2, this law provides quite bad results for most fiber fractions and other models should better be employed. Among them, the so-called dit horizontal model

$$\frac{1}{E_c^T} = \frac{\sqrt{V_f}}{E_f \sqrt{V_f} + E_m (1 - \sqrt{V_f})} + \frac{1 - \sqrt{V_f}}{E_m} \tag{3}$$

and the vertical model

$$E_c^T = \frac{E_f E_m}{E_m + E_f (1 - \sqrt{V_f}) / \sqrt{V_f}} + (1 - \sqrt{V_f}) E_m \tag{4}$$

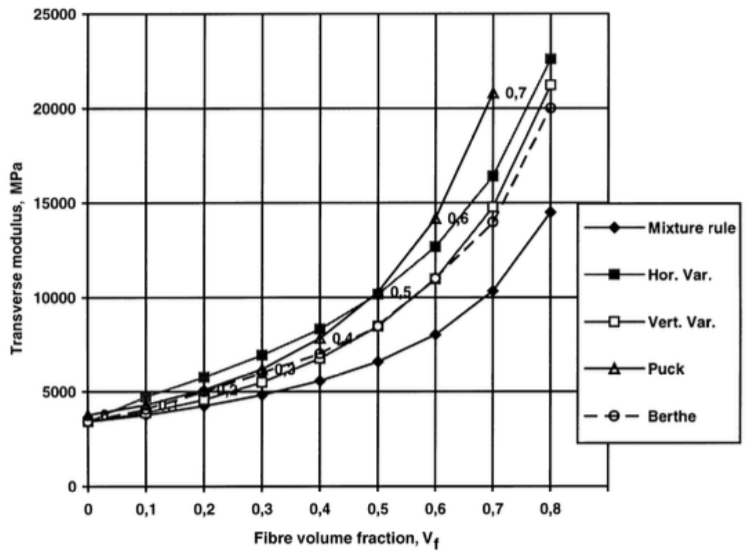


Figure 1: Estimations of the transverse modulus of a UD in terms of the fiber volume fraction.

# 2 Linear Elasticity

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \quad \Longleftrightarrow \quad \forall (i,j) \qquad \sigma_{ij} = C_{ijkl}\varepsilon_{lk} \tag{5}$$

## 2.1 Elasticity Matrix

The elasticity tensor, after accounting for the various symmetries, exhibits 21 independent components that enable to describe the most general anisotropic behavior of a material. They result from both the symmetry of the stress and strain tensors and of the so-called Onsager symmetries due to the properties of the elastic strain energy density.

It therefore can be represented by a symmetric $(6 \times 6)$ matrix.

### 2.1.1 Vectorial notations

To do so, on uses the vectorial notation for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ that accounts for the symmetry of those tensors. They exhibit 6 independant components that have to be ordered according to a certain logic.

In the scope of this course, we'll adopt the following convention:
- stress vector:

$$\begin{aligned} {}^T\{\sigma\} &= <\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{31}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{12}> \\ &= <\sigma_1, \sigma_2, \sigma_3, \sqrt{2}\sigma_4, \sqrt{2}\sigma_5, \sqrt{2}\sigma_6> \end{aligned}$$

- strain vector

$$\begin{aligned} {}^T\{\varepsilon\} &= <\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\varepsilon_{31}, \sqrt{2}\varepsilon_{23}, \sqrt{2}\varepsilon_{12}> \\ &= <\varepsilon_1, \varepsilon_2, \varepsilon_3, \sqrt{2}\varepsilon_4, \sqrt{2}\varepsilon_5, \sqrt{2}\varepsilon_6> \end{aligned}$$

### 2.1.2 Explanation of this notation: strain energy density $W_d$

$$2W_d = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = tr(\boldsymbol{\sigma} \cdot^T \boldsymbol{\varepsilon}) = \sum_{i,j} \sigma_{ij}\varepsilon_{ij}$$

$$2W_d = \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\left[\sigma_{31}\varepsilon_{31} + \sigma_{23}\varepsilon_{23} + \sigma_{12}\varepsilon_{12}\right]$$

$$\Longrightarrow 2W_d =^T \{\sigma\} \cdot \{\varepsilon\}$$

Remark: This choice is not unique and is not the one generally retained. We'll find more often
- stress vector:

$${}^T\{\hat{\sigma}\} = <\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{31}, \sigma_{23}, \sigma_{12}>$$

- strain vector

$${}^T\{\hat{\varepsilon}\} = <\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{31}, 2\varepsilon_{23}, 2\varepsilon_{12}>$$

This complexifies the base changes techniques discussed after and looses the symmetry of the problem. Nevertheless, the shape of the constitutive equations we'll discuss here will be slightly different of the classical one.

### 2.1.3 Frame change formula

Case of a rotation of angle $\theta$ around the material axis $\vec{N}_3$. The rotation matrix is:

$$[P] = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $C = \cos\theta$ et $S = \sin\theta$.

For a (3,3) symmetric matrix $\boldsymbol{A}$ with coefficients $A_{ij}$, it can be shown that:

$$\left\{\hat{A}\right\}_{(x,y,z)} = [T]\left\{\hat{A}\right\}_{(N_1,N_1,N_2)} \tag{6}$$

where

$$[T] = \begin{bmatrix} C^2 & S^2 & 0 & 0 & 0 & -\sqrt{2}SC \\ S^2 & C^2 & 0 & 0 & 0 & +\sqrt{2}SC \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & S & C & 0 \\ \sqrt{2}SC & \sqrt{2}SC & 0 & 0 & 0 & C^2 - S^2 \end{bmatrix}$$

### 2.1.4 Matrix form of the constitutive equations

The 21 coefficients of the elasticity tensor enable to transform the tensorial realtion into a more convenient matricial form relating $\{\hat{\sigma}\}$ and $\{\hat{\varepsilon}\}$:

$$\{\hat{\sigma}\} = \left[\hat{C}\right]\{\hat{\varepsilon}\} \tag{7}$$

So

$$\left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{31} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{12} \end{array}\right\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \mathrm{SYM} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \left\{\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{31} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{12} \end{array}\right\} \tag{8}$$

### 2.1.5 Frame change une the constitutive equations

Change from frame $\mathcal{R}_0$ to frame $\mathcal{R}_1$, characterized by an orthogonal matrix $[P]$ associated to a matrix $[T]$ for stress and strain vectors.

This leads to

$$\left[\hat{C}_1\right] = [T]^{-1}\left[\hat{C}_0\right][T]$$

## 2.2 Orthotropic Material

### 2.2.1 Compliance matrix

$$\left\{\begin{array}{c}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \sqrt{2}\varepsilon_{31}\\ \sqrt{2}\varepsilon_{23}\\ \sqrt{2}\varepsilon_{12}\end{array}\right\} = \left[\begin{array}{cccccc} 1/E_1 & -\nu_{12}/E_1 & -\nu_{13}/E_1 & 0 & 0 & 0\\ -\nu_{21}/E_2 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0\\ -\nu_{31}/E_3 & -\nu_{32}/E_3 & 1/E_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/(2G_{23}) & 0 & 0\\ 0 & 0 & 0 & 0 & 1/(2G_{13}) & 0\\ 0 & 0 & 0 & 0 & 0 & 1/(2G_{12})\end{array}\right] \left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{31}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{12}\end{array}\right\} \tag{9}$$

### 2.2.2 Stiffness matrix

The previous law, written in terms of meaningful physical parameters, can also be inverted.

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{31}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{12}\end{array}\right\} = \left[\begin{array}{cccccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \sqrt{2}\varepsilon_{31}\\ \sqrt{2}\varepsilon_{23}\\ \sqrt{2}\varepsilon_{12}\end{array}\right\} \tag{10}$$

with the following relations:

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} \quad ; \quad C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} \quad ; \quad C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta}$$

$$C_{12} = \frac{\nu_{21}-\nu_{31}\nu_{23}}{E_2E_3\Delta} \quad ; \quad C_{13} = \frac{\nu_{31}-\nu_{21}\nu_{32}}{E_2E_1\Delta} \quad ; \quad C_{23} = \frac{\nu_{32}-\nu_{12}\nu_{31}}{E_1E_3\Delta}$$

$$C_{44} = 2G_{23} \quad ; \quad C_{55} = 2G_{13} \quad ; \quad C_{66} = 2G_{12}$$

$$\Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

## 2.3 Transversely isotropic material

This case can be easily understood from the former case. The material axe of transverse isotropy is assumed to be $\vec{e}_1$.

### 2.3.1 Compliance matrix

$$\left\{\begin{array}{c}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \sqrt{2}\varepsilon_{31}\\ \sqrt{2}\varepsilon_{23}\\ \sqrt{2}\varepsilon_{12}\end{array}\right\} = \left[\begin{array}{cccccc}1/E_1 & -\nu_{12}/E_1 & -\nu_{12}/E_1 & 0 & 0 & 0\\ -\nu_{21}/E_2 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0\\ -\nu_{21}/E_1 & -\nu_{23}/E_2 & 1/E_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1+\nu_{23}}{E_2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}})\end{array}\right] \left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{31}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{12}\end{array}\right\} \tag{11}$$

### 2.3.2 Stiffness matrix

Similar to the orthotropic case, only the calculations of the coefficients change.

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_2\Delta} \quad ; \quad C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_2\Delta} \quad ; \quad C_{33} = C_{22}$$

$$C_{12} = \frac{\nu_{21}-\nu_{21}\nu_{23}}{E_2E_2\Delta} \quad ; \quad C_{13} = C_{12} \quad ; \quad C_{23} = \frac{\nu_{32}-\nu_{12}\nu_{21}}{E_1E_2\Delta}$$

$$C_{44} = C_{22} - C_{23} = \frac{E_2}{1+\nu_{23}} \qquad ; \qquad C_{55} = C_{66} = 2G_{12}$$

$$\Delta = \frac{1-2\nu_{12}\nu_{21}-\nu_{23}\nu_{32}--2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2^2}$$

# 3 Orthotropic plate under plane stress assumption

$$\left\{\begin{array}{c}\sigma_1\\ \sigma_2\\ \sqrt{2}\sigma_6\end{array}\right\} = \left[\begin{array}{ccc}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{array}\right] \left\{\begin{array}{c}\varepsilon_1\\ \varepsilon_2\\ \sqrt{2}\varepsilon_6\end{array}\right\} \tag{12}$$

Relation with the properties obtained from experiments.

$$Q_{11} = \frac{E_L}{1-\nu_{LT}\nu_{TL}} = \frac{E_L}{1-\nu_{LT}^2\frac{E_T}{E_L}} \tag{13}$$

$$Q_{22} = \frac{E_T}{1-\nu_{LT}\nu_{TL}} = \frac{E_T}{1-\nu_{LT}^2\frac{E_T}{E_L}} = \frac{E_T}{E_L}Q_{11} \tag{14}$$

$$Q_{12} = \frac{\nu_{LT}E_T}{1-\nu_{LT}\nu_{TL}} = \nu_{LT}Q_{22} \tag{15}$$

$$Q_{66} = 2G_{LT} \tag{16}$$

See formular for the calculation of this matrix when the material axes are not aligned with the global frame.