Constraint Programming: Examples

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Objectives

- to illustrate constraints by examples
- to give an intuitive taste of several **types of constraint**
- to illustrate **constrained problems** by examples
- to give an intuitive taste of **modelling**
- to give an intuitive notion of CSP

Examples of constraints

Constraints : intuitively

- a relation between objects (represented by variables)
- a constraint can specify :
  - partial, incomplete information

    « the captain is at least 40 year old »
  - fuzzy information

    « the captain is about 40 year old »
- a constraint is declarative (independant from the operationnal process)
- a constraint is not oriented (relation) :

  $x + y = z$: if x and y are known, we determine z ; if x and z are known we determine y, ...
- the order to set constraints does not influence the semantics (but generally solving efficiency)

Numeric constraints

Atomic constraints :

$$x^2 = 2 \tag{1}$$

$\Rightarrow$ computation domains of variables must be known :

- x rationnal number : no solution to (1)
- x real numbers : two solutions $\{-\sqrt{2}, \sqrt{2}\}$

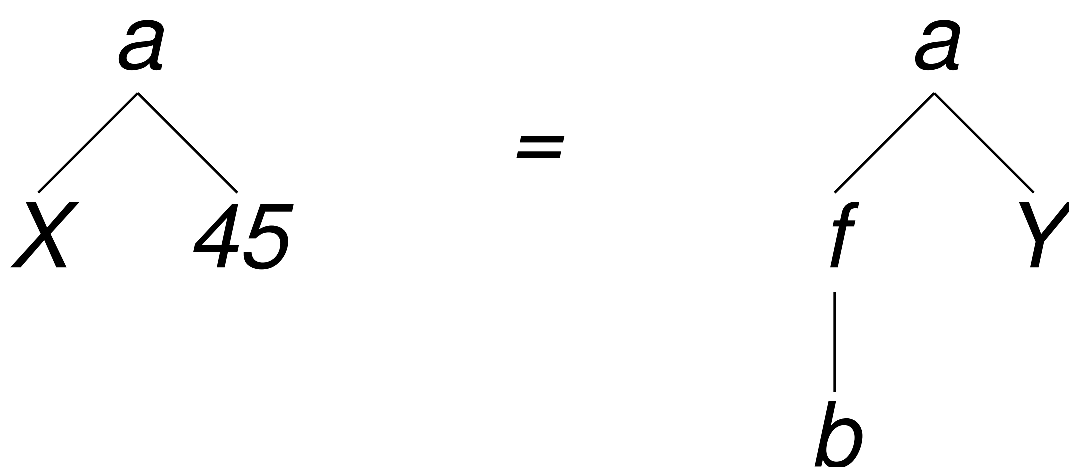
More generally :

$$x^2 - y = 0 \wedge x^2 + y^2 = 1$$

Conjunctions, disjunctions, negations of atomic constraints

Constraints over trees

$$a(X, 45) = a(f(b), Y)$$



$\Rightarrow X = f(b)$ and $Y = 45$

The equality relation is not oriented

Boolean constraints

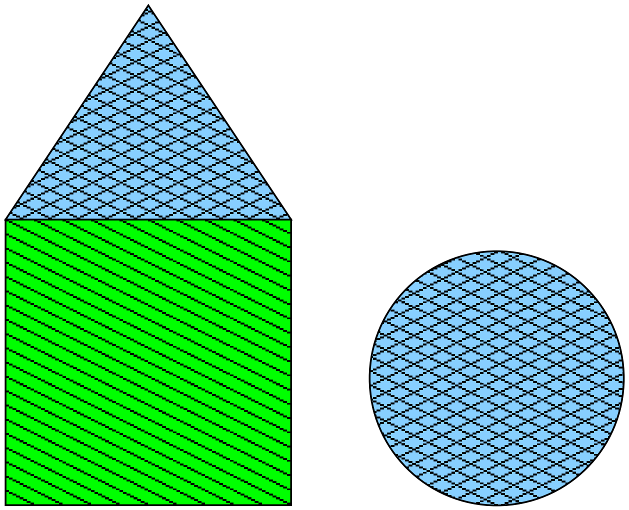
- variables are **true** or **false**
  (or often 0 or 1)
- constraints represent Boolean operators
  (as relations, thus, the result is a variable)

  $Z =$ **X wedge Y** is represented by the constraint :
  - $Z \iff X \wedge Y$
  - or $and(X, Y, Z)$

Symbolic constraints

World of blocs :

$$\textit{blue}(X) \wedge \textit{on}(X, Y)$$



$\Rightarrow X =$ triangle and $Y =$ rectangle

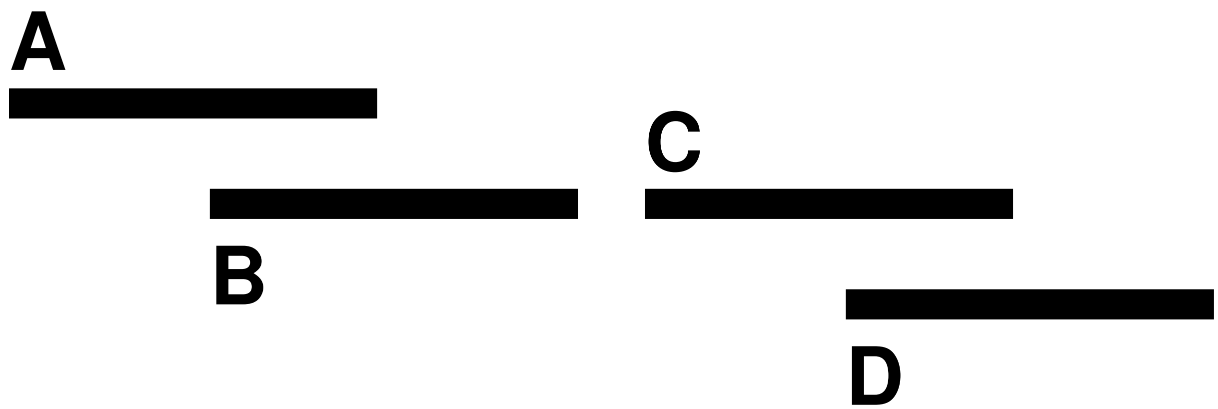
Qualitative temporal reasonning

Allen temporal logic :

allen(AoverlapB, BbeforeC, R_AC)

$\wedge$ *allen(BbeforeC,CoverlapD, R_BD)*

$\wedge$ *allen(R_AC, CoverlapD, R_AD)*



$\Rightarrow$ {
R_AC= AbeforeC
and R_BD=BbeforeD
and R_AD=AbeforeD

Global constraints

- *atmost*$(2, [X_1, X_2, X_3, X_4, X_5], 1)$
  at most two variables among $\{X_1, X_2, X_3, X_4, X_5\}$ are equal to 1
- *alldifferent*$([X_1, X_2, X_3, X_4, X_5])$
  the variables $\{X_1, X_2, X_3, X_4, X_5\}$ are pair-wise distinct

Examples of problems

Constraint Satisfaction Problems (CSP)

Given :

- some type of variables to represent objects
- domains over which variables can range
- some types constraints to set relation between objects

Formulate your problem as a CSP :

- a set of variables together with their initial domains
- a set of constraints linking your variables (objects)

DONALD + GERALD = ROBERT (1/3)

- cryparithmetic problem over integers
- replace each letter by a different digit such that

	D	O	N	A	L	D
+	G	E	R	A	L	D
=	R	O	B	E	R	T

is a correct sum

DONALD + GERALD = ROBERT (2/3)

First modelling :

- variables : $D, O, N, A, L, G, E, R, B, T$
- integer domains :
  $[1..9]$ for D and G
  $[0..9]$ for O, N, A, L, E, R, B, T
- constraint :

$$100000.D + 10000.O + 1000.N + 100.A + 10.L + D$$
$$+ 100000.G + 10000.E + 1000.R + 100.A + 10.L + D$$
$$= 100000.R + 10000.O + 1000.B + 100.E + 10.R + T$$

DONALD + GERALD = ROBERT (3/3)

Second modelling : use of carry variables

- variables : $D, O, N, A, L, G, E, R, B, T, C_1, C_2, C_3, C_4, C_5$
- integer domains :
  $[1..9]$ for D and G
  $[0..9]$ for O, N, A, L, E, R, B, T
  $[0..1]$ for C_1, C_2, C_3, C_4, C_5
- constraint :

$$2.D = 10.C_1 + T$$
$$2.L + C_1 = 10.C_2 + R$$
$$2.A + C_2 = 10.C_3 + E$$
$$N + R + C_3 = 10.C_4 + B$$
$$O + E + C_4 = 10.C_5 + O$$
$$D + G + C_5 = R$$

Zebra puzzle (1/10)

1 A small street is composed of 5 colored houses.

2 Five men of different nationalities live in these five houses.

3 Each man has a different profession.

4 Each man likes a different drink.

5 Each man has a different pet animal.

Zebra puzzle (2/10)

6. The Englishman lives in the red house.
7. The Spaniard has a dog.
8. The Japanese is a painter.
9. The Italian drinks tea.
10. The Norwegian lives in the first house on the left.
11. The owner of the green house drinks coffee.
12. The green house is on the right of the white house.
13. The sculptor breeds snails.
14. The diplomat lives in the yellow house.
15. They drink milk in the middle house.
16. The Norwegian lives next door to the blue house.
17. The violonist drinks fruit juice.
18. The fox is in the house next to the doctor's.
19. The horse is in the house next to the diplomat's.

Who has the zebra and who drinks water ?

Zebra puzzle (3/10)

1. A small street is composed of 5 colored houses.
2. Five men of different nationalities live in these five houses.
3. Each man has a different profession.
4. Each man likes a different drink.
5. Each man has a different pet animal.

Zebra puzzle (4/10)

- Idea 1 : men numbered from 1 to 5

  ⇒ english=3 means the 3rd man is english

  ⇒ what to do with the constraint :
  "The green house is on the right of the white house."

- Idea 2 : houses numbered from 1 to 5

  ⇒ englishman=3 means the Englishman lives in the 3rd house

  ⇒ yellow=2 means the 2nd house is yellow

  - "The green house is on the right of the white house."
    ⇒ green = white + 1

  ⇒ all constraints of the puzzle can be used

Zebra puzzle : variables (5/10)

Determining variables :

6. The Englishman lives in the red house.
7. The Spaniard has a dog.
8. The Japanese is a painter.
9. The Italian drinks tea.
10. The Norwegian lives in the first house on the left.
11. The owner of the green house drinks coffee.
12. The green house is on the right of the white house.
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16. The Norwegian lives next door to the blue house.
17. The violonist drinks fruit juice.
18. The fox is in the house next to the doctor's.
19. The horse is in the house next to the diplomat's.

Zebra puzzle (6/10)

Variables : 25 (5x5)

- men : englishman, spaniard, japanese, italian, norwegian
- profession : painter, sculptor, diplomat, violonist, doctor
- drink : tea, coffee, milk, juice, ? ? ?
- pet animal : dog, snail, fox, horse, ? ? ?
- colour : red, green, white, yellow, blue

two variables are missing ? ? ?

Zebra puzzle (7/10)

There is some more information in the query :
Who has the zebra and who drinks water ?

Variables : 25 (5x5)

- men : englishman, spaniard, japanese, italian, norwegian
- profession : painter, sculptor, diplomat, violonist, doctor
- drink : tea, coffee, milk, juice, **water**
- pet animal : dog, snail, fox, horse, **zebra**
- colour : red, green, white, yellow, blue

Domains : [1..5] (5 houses)

Zebra puzzle : constraints (8/10)

1. A small street is composed of 5 colored houses
   `all_different`(red, green, white, yellow, blue)
2. Five men of different nationalities live in these five houses.
   `all_different`(englishman, spaniard, japanese, italian, norwegian)
3. Each man has a different profession.
   `all_different`(painter, sculptor, diplomat, violonist, doctor)
4. Each man likes a different drink.
   `all_different`(tea, coffee, milk, juice, water)
5. Each man has a different pet animal.
   `all_different`(dog, snail, fox, horse, zebra)

Domains : [1..5]

Zebra puzzle : constraints (ctd) (9/10)

6	The Englishman lives in the red house.	englishman=red
7	The Spaniard has a dog.	spaniard=dog
8	The Japanese is a painter.	japanese=painter
9	The Italian drinks tea.	italian=tea
10	The Norwegian lives in the first house on the left.	norwegian=1
11	The owner of the green house drinks coffee.	green=coffee
12	The green house is on the right of the white house.	green=white+1
13	The sculptor breeds snails.	sculptor=snail
14	The diplomat lives in the yellow house.	diplomat=yellow
15	They drink milk in the middle house.	milk=3
16	The Norwegian lives next door to the blue house.	|norwegian - blue| = 1
17	The violonist drinks fruit juice.	violonist = juice
18	The fox is in the house next to the doctor's.	|fox - doctor|=1
19	The horse is in the house next to the diplomat's.	|horse - diplomat| =1

Zebra puzzle : solution (10/10)

Solution :

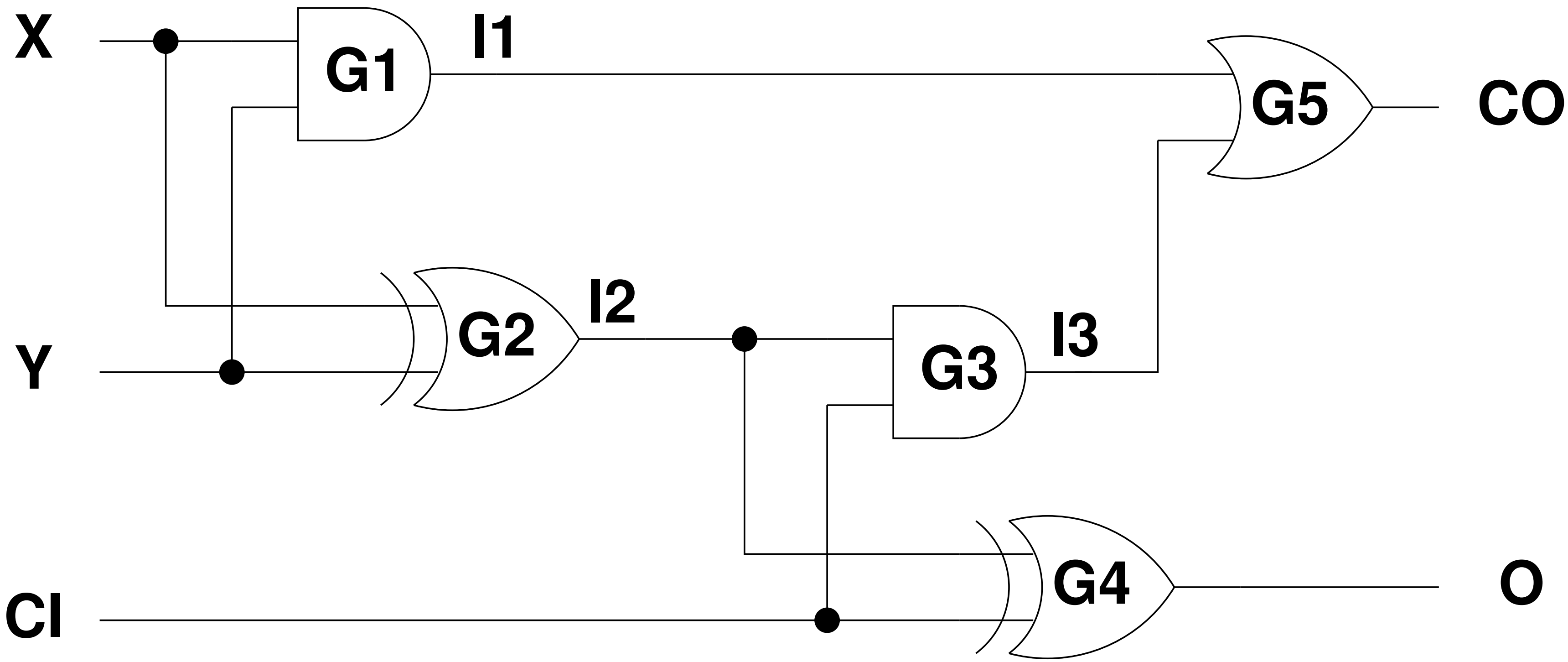
- Englishman=3, Spaniard=4, Japanese=5, Italian=2, Norwegian=1
- Tea=2, Coffee=5, Milk=3, Juice=4, Water=1
- Red=3, Green=5, White=4, Yellow=1, Blue=2
- Painter=5, Sculptor=3, Diplomat=1, Violonist=4, Doctor=2
- Dog=4, Snail=3, Fox=1, Horse=2, Zebra=5

Thus,

- the Japanese has the zebra,
- and the Norwegian drinks water.

Full-adder (1)

Full-adder :



Full-adder (2)

Modelling :

- **Variables** : inputs/outputs of gate
- **Domains** : [0,1]
- **Boolean constraints** :
  - $(I1 \iff X \wedge Y),\ (I2 \iff X \oplus Y),\ (I3 \iff I2 \wedge CI)$ $(O \iff I2 \oplus CI),\ (CO \iff I1 \vee I3)$
  - or

    $and(X, Y, I1),\ \ xor(X, Y, I2),\ \ and(I1, CI, I3)$
    $xor(I2, CI, O),\ \ or(I1, I3, CO)$

n-Queens

Place n queens on a $n \times n$ board so that they do not attack each other

Modelling :

- **Variables** : $c_1, \ldots, c_n$

  one per column : the value of c_i represents the line where the queen is in the column

- **Domains** : [1..n]

- **Constraints** : for $i \in [1..n-1]$ and $j \in [i+1..n]$
  - not two queens on the same line : $c_i \neq c_j$
  - not 2 queens on the same SW-NE diagonal : $c_i \neq c_j + j - i$
  - not 2 queens on the same NW-SE diagonal : $c_i \neq c_j + i - j$