Constraint Programming
– Filtering : Part 1 –

Christophe Lecoutre
lecoutre@cril.fr

CRIL-CNRS UMR 8188
Universite d'Artois
Lens, France

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Outline

1. Filtering Domains with Constraints
2. Principle of Constraint Propagation

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Filtering Domains by means of Constraints

Each constraint represents a "sub-problem" from which some *inconsistent* values can be deleted.

Inconsistent values belong to no solution (of the sub-problem).

Several levels/types of filtering can be defined. For the moment, we only cite:

- AC (Arc Consistency): all inconsistent values are identified and deleted
- BC (Bounds Consistency): inconsistent values corresponding to the bounds of the domains are identified and deleted

Warning.

For non-binary constraints, AC is often denoted by GAC (but not in this course).

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Example.

Constraint $x < y$ with

- $dom(x) = 10..20$
- $dom(y) = 0..15$

After AC filtering, we obtain:

- $dom(x) = 10..14$
- $dom(y) = 11..15$

After BC filtering, we obtain:

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Constraint $w + 3 = z$ with

- $dom(w) = \{1, 3, 4, 5\}$
- $dom(z) = \{4, 5, 8\}$

After AC filtering, we obtain:

- $dom(w) = \{1, 5\}$
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Notion of Support

For a constraint c

- an allowed tuple, or tuple accepted by c, is an element of $A = rel(c)$
- a valid tuple is an element of $V = \Pi_{x \in scp(c)} dom(x)$
- a support (on c) is a tuple that is both allowed and valid, i.e., an element of $A \cap V$

Remark.
A support on c is what we have previously informally called a solution of the "sub-problem" c.

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Example.

Let c_{xyz} be a ternary constraint, and let us suppose that $dom(x) = dom(y) = \{a, b\}$ and $dom(z) = \{b, c\}$. We have:

- $A = rel(c_{xyz})$
- $V = dom(x) \times dom(y) \times dom(z)$

A
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a b b
a c c
b a a
b b b
c a a
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(z, c) has no support ✗

Arc Consistency (AC)

Definition

A constraint c is arc-consistent (AC) iff $\forall x \in scp(c)$, $\forall a \in dom(x)$, there exists a support of (x, a) on c, i.e., a support τ on c such that $\tau[x] = a$.

Example.

Let x and y be two variables such that $dom(x) = dom(y) = \{1, 2\}$, and let $x = y$ be a binary constraint.

- the tuple $\tau = (1, 2)$ $\quad // \ \tau[x] = 1 \wedge \tau[y] = 2$
  - is valid
  - but not accepted by $x = y$
- the tuple $\tau = (3, 3)$
  - is not valid,
  - but accepted by $x = y$
- the tuple $\tau = (2, 2)$
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  it represents a support of both $(x, 2)$ and $(y, 2)$ on $x = y$

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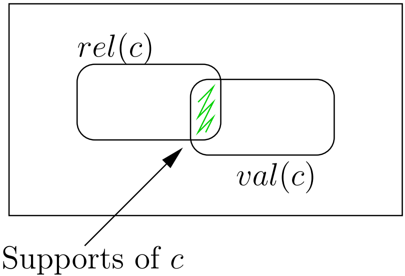
Supports

In other words, the supports on a constraint c are those tuples that are present in the intersection of :

- the set of allowed tuples: $rel(c)$
- the set of valid tuples: $val(c) = \Pi_{x \in scp(c)} dom(x)$



⇒ We need to "identify" these supports for filtering

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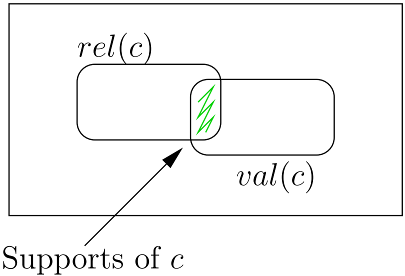

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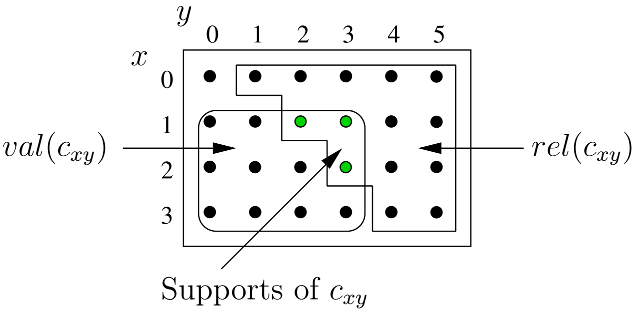

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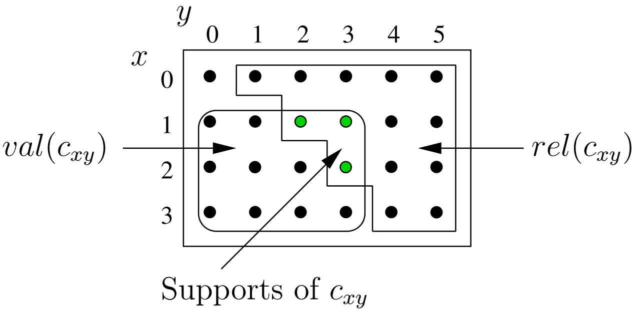

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AC Algorithm

Definition

A value (x, a) is *arc-inconsistent* on a constraint c when there is no support of (x, a) on c.

Definition

An AC algorithm for a constraint c is an algorithm that removes all values that are arc-inconsistent on c; the algorithm is said to enforce/establish AC on c.

Here is an AC algorithm that can be used in theory with any constraint c.

Algorithm 1: filterAC(c: Constraint)

```
for each variable x ∈ scp(c) do
    for each value a ∈ dom(x) do
        if ¬ seekSupport(c, x, a)    // function to be implemented
        then
            remove a from dom(x)
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Algorithm 2: filterAC(c: Constraint)

for *each variable* $x \in scp(c)$ **do**
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AC Filtering for `allDifferent`

Proposition

A constraint `allDifferent`*(X) is AC iff* $\forall X' \subseteq X$,

$$|dom(X')| = |X'| \Rightarrow \forall x \in X \setminus X', dom(x) = dom(x) \setminus dom(X')$$

where $dom(X') = \cup_{x' \in X'} dom(x')$

Remark.

A subset X' of variables such that $|dom(X')| = |X'|$ is called a Hall set.

Example.

The set of variables $\{x, y, z\}$ such that:

- $dom(x) = \{a, b\}$,
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is a Hall set (of size 3).

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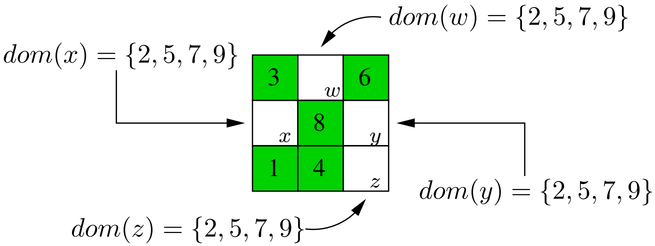

Can we filter?

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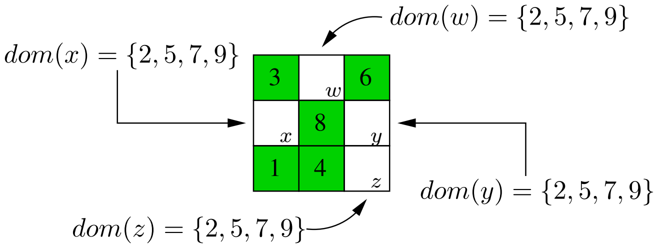

Can we filter?

Identification of Hall sets

The same constraint as previously, but variables have different domains.

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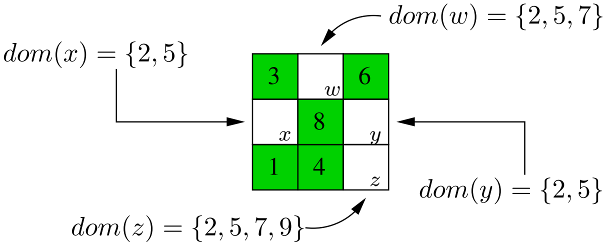

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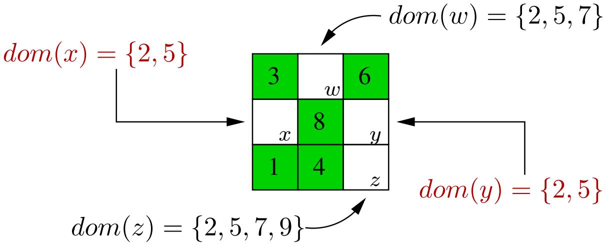

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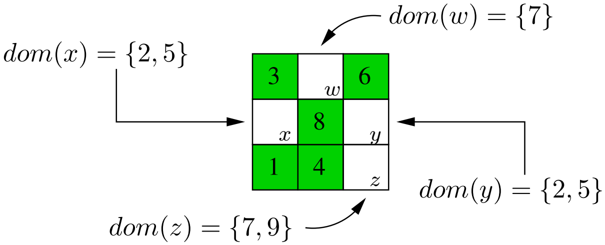

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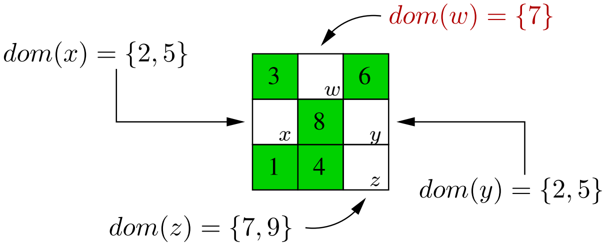

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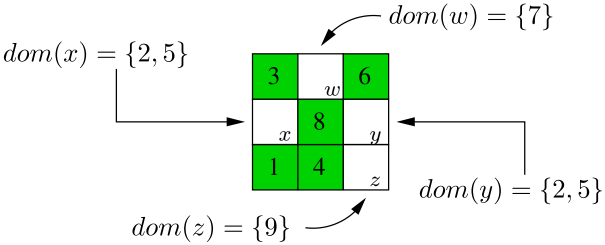

AC Filtering for `cardinality`

Definition

A constraint `cardinality`(X, V, L, U) forces the variables in X to take their values in V with the restriction that each value v_i in V is assigned at least $L(v_i)$ times and at most $U(v_i)$ times.

Example.

Three sets:

- $Agents = \{Peter, Paul, Mary, John, Bob, Mike, Julia\}$
- $Days = \{Monday, Tuesday, ..., Sunday\}$
- $Activities = \{M(orning), D(ay), N(ight), B(ackup), O(ff)\}$.
- We want a roster that looks like:

	Mo	Tu	We	Th	Fr	Sa	Su
Peter	D	N	N	N	O	O	O
Paul	O	O	D	D	M	M	B
Mary	M	M	D	D	O	O	N
...							

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For simplicity, we only reason here on Monday. Our variables X represent the agents, and we have $\forall x \in X, dom(x) = \{M, D, N, B, O\}$.
The constraint `cardinality`(X,$\{M, D, N, B, O\}$,L,U) is such that:

- $L = \{1, 1, 1, 0, 0\}$
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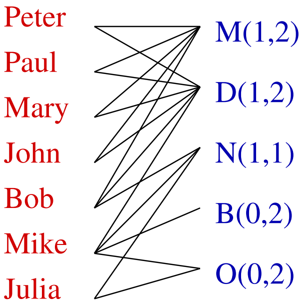

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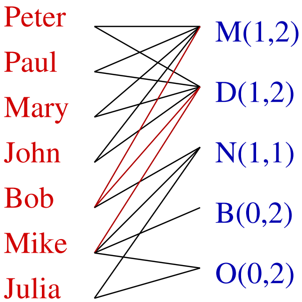

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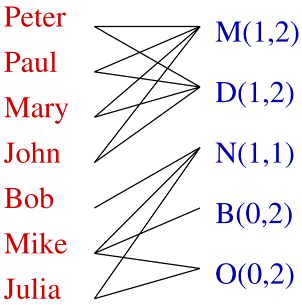

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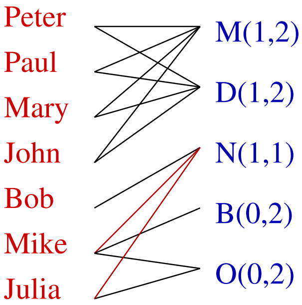

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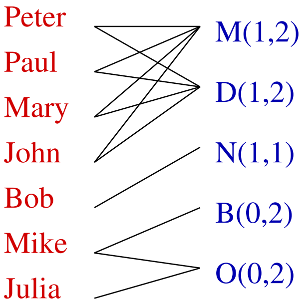

AC Filtering for sum : $\sum_{i=1}^{r} c_i x_i \geq L$

Domains of variables
w, x, y and z

dom			
w	x	y	z
1	1	2	2
2	2	3	3
3	3	4	4

Constraint c_{wxyz} : $w + 2x + 4y + 5z \geq 42$

Domains of variables
w, x, y et z
after AC filtering of c_{wxyz}

dom			
w	x	y	z
1	~~1~~	~~2~~	~~2~~
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Complexity?

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w, x, y and z

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		2	2

Constraint $c_{wxyz} : w + x + y + z \neq 5$

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Complexity?

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Constraint $c_{wxyz} : 82 \geq 27w + 37x + 45y + 53z \geq 80$

Domains of variables w, x, y and z after AC filtering of c_{wxyz}

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- construction of a graph (Knapsack)
- reduction of the graph
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Warning.

Pseudo-polynomial Complexity $O(rU^2)$

Example.

Illustration of this approach with:

- the constraint $12 \geq 2x_1 + 3x_2 + 4x_3 + 5x_4 \geq 10$
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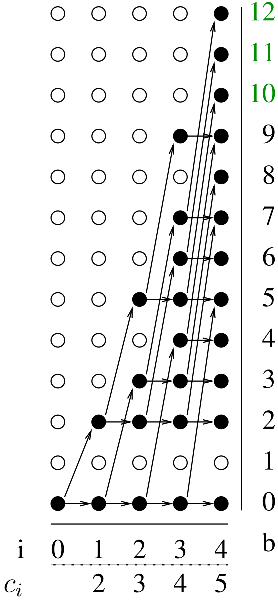
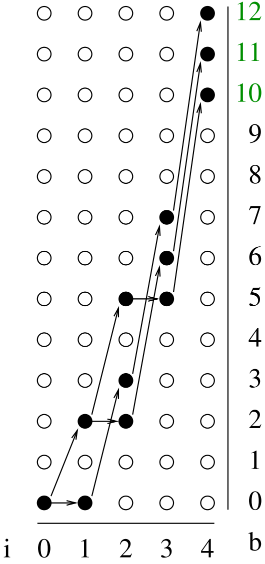
Example

Knapsack Graph





Reduced Knapsack Graph

AC Filtering for `or` (meta-constraint)

Constructive Disjunction

Enforcing AC on a meta-constraint $\texttt{or}(c_1,c_2)$ can be achieved by *constructive disjunction*: for each variable x, $dom(x)$ is the union of the domains of x obtained after AC filtering on c_1 and AC filtering on c_2.

Example.

Let x be a variable such that $dom(x) = \{1, 2, 3\}$ and the meta-constraint $\texttt{or}(x = 1, x = 2)$.

AC on $x = 1$ *yields* $dom^1(x) = \{1\}$
AC on $x = 2$ *yields* $dom^2(x) = \{2\}$

AC on $\texttt{or}(x = 1, x = 2)$ reduces $dom(x)$ to $dom^1(x) \cup dom^2(x) = \{1, 2\}$

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AC Filtering for and (meta-constraint)

Proposition

AC on the conjunction `and`(c_1, c_2) *is with respect to AC enforced independently on* c_1 *and* c_2*:*

- *generally stronger,*
- *equivalent when* $|scp(c_1) \cap scp(c_2)| \leq 1$

Example

Let x and y two variables such that $dom(x) = dom(y) = \{1, 2, 3\}$ and the meta-constraint `and`$(x \neq y, x \leq y)$.

- AC on $x \neq y$ as well as AC on $x \leq y$ have no effect
- AC on `and`$(x \neq y, x \leq y)$ permits to have:
  - $dom(x)$ reduced to $\{1, 2\}$
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Outline

1. Filtering Domains with Constraints
2. Principle of Constraint Propagation

Constraint Propagation

Definition

A constraint network P is AC iff each constraint of P is AC.

Definition

Computing the AC-closure of a constraint network P is the fact of removing all arc-inconsistent of P (when considering any constraint of P).

May we sollicit each constraint once (for filtering) in order to compute the AC-closure?

NO because when some values are filtered out by a constraint, this can give new opportunities to other constraints to filter again.

The process that involves executing filtering operations, by solliciting constraints in turn, until a fixed point is reached is called **constraint propagation**.

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Constraint Propagation Algorithm

Algorithm 4: constraintPropagationOn(P: CN): Boolean

```
Q ← ctrs(P)
while Q ≠ ∅ do
    pick and delete c from Q
    X_evt ← c.filter()   // X_evt denotes the set of variables with reduced domains (after filtering by means of c)
    if ∃x ∈ X_evt such that dom(x) = ∅ then
        return false        // global inconsistency detected
    foreach c' ∈ ctrs(P) such that c' ≠ c and X_evt ∩ scp(c') ≠ ∅ do
        add c' to Q
return true
```

Remark.

If each call $c.filter()$ enforces AC on c, then the algorithm computes the AC-closure of P.

Constraint Propagation Algorithm

Algorithm 5: constraintPropagationOn(P: CN): Boolean

$Q \leftarrow ctrs(P)$
while $Q \neq \emptyset$ **do**
    pick and delete c from Q
    $X_{evt} \leftarrow c.filter()$ // X_{evt} denotes the set of variables with reduced domains (after filtering by means of c)
    if $\exists x \in X_{evt}$ *such that* $dom(x) = \emptyset$ **then**
        return *false* // global inconsistency detected
    foreach $c' \in ctrs(P)$ *such that* $c' \neq c$ *and* $X_{evt} \cap scp(c') \neq \emptyset$ **do**
        add c' to Q
return *true*

Remark.

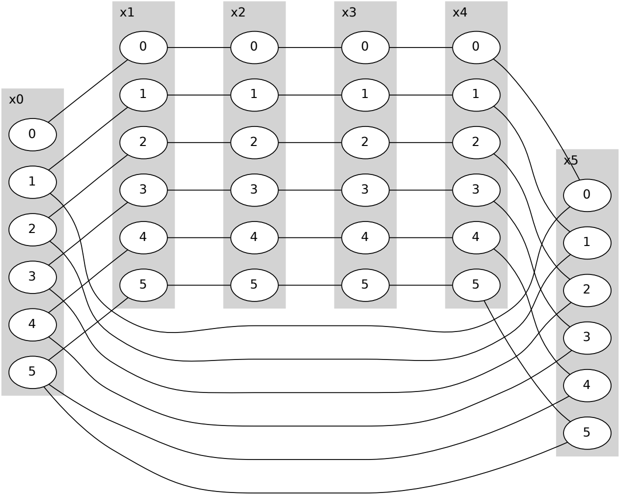
If each call $c.filter()$ enforces AC on c, then the algorithm computes the AC-closure of P.

Domino Problem

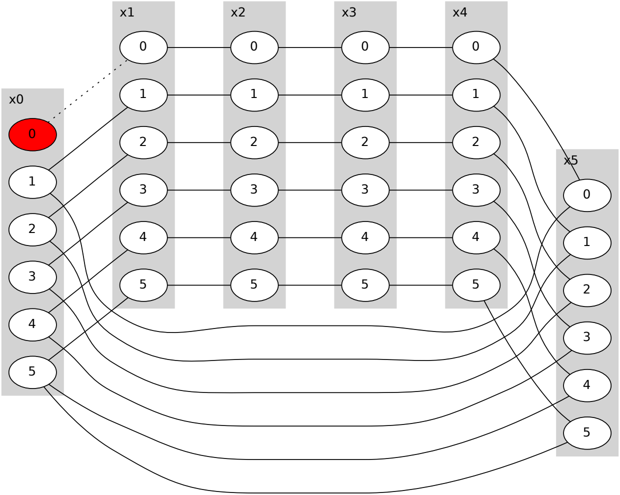
The instance `domino-6` is represented by the following CN P:

- $vars(P) = \{$
  x_0 with $dom(x_0) = \{0, 1, 2, 3, 4, 5\}$,
  x_1 with $dom(x_1) = \{0, 1, 2, 3, 4, 5\}$,
  x_2 with $dom(x_2) = \{0, 1, 2, 3, 4, 5\}$,
  x_3 with $dom(x_3) = \{0, 1, 2, 3, 4, 5\}$,
  x_4 with $dom(x_4) = \{0, 1, 2, 3, 4, 5\}$,
  x_5 with $dom(x_5) = \{0, 1, 2, 3, 4, 5\}$
  $\}$
- $ctrs(P) = \{$
  $x_0 = x_1$,
  $x_1 = x_2$,
  $x_2 = x_3$,
  $x_3 = x_4$,
  $x_4 = x_5$,
  $(x_0 = x_5 + 1 \wedge x_0 < 5) \vee (x_0 = x_5 \wedge x_0 = 5)$
  $\}$

Constraint Propagation on `domino-6`



Constraint Propagation on `domino-6`



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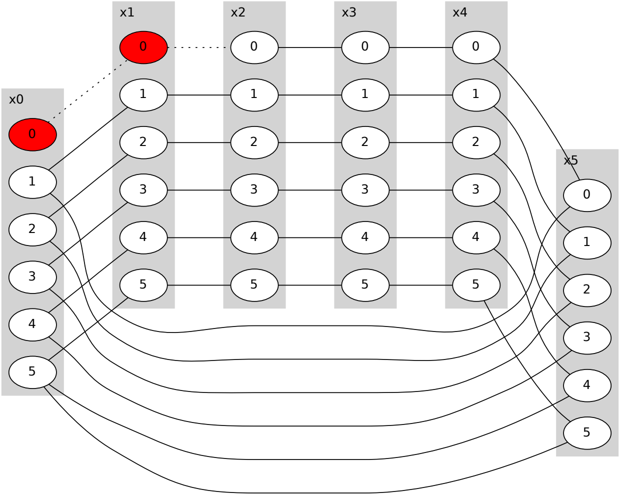
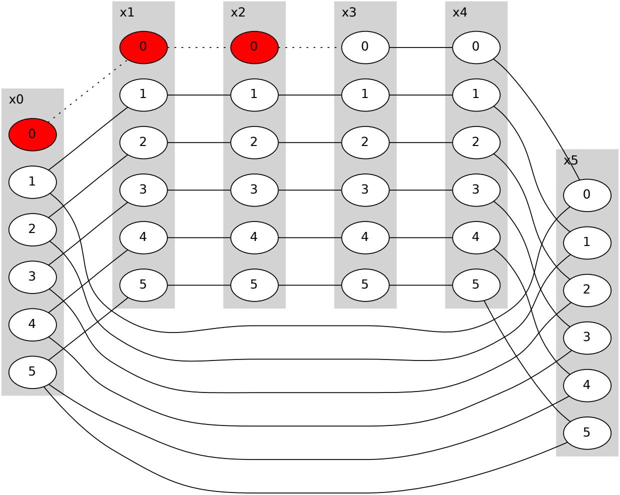
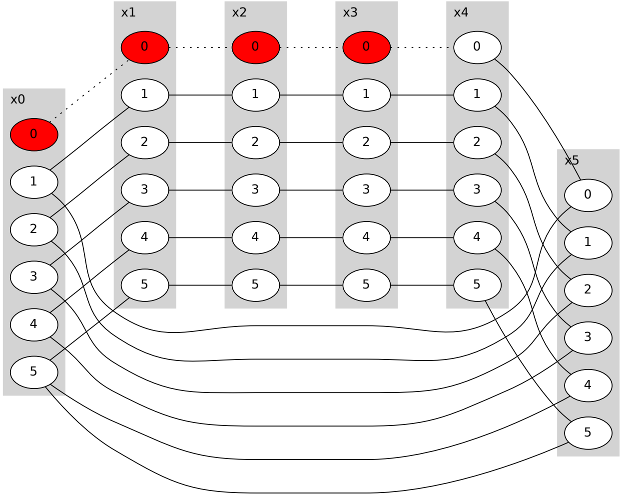


Constraint Propagation on domino-6



Constraint Propagation on domino-6



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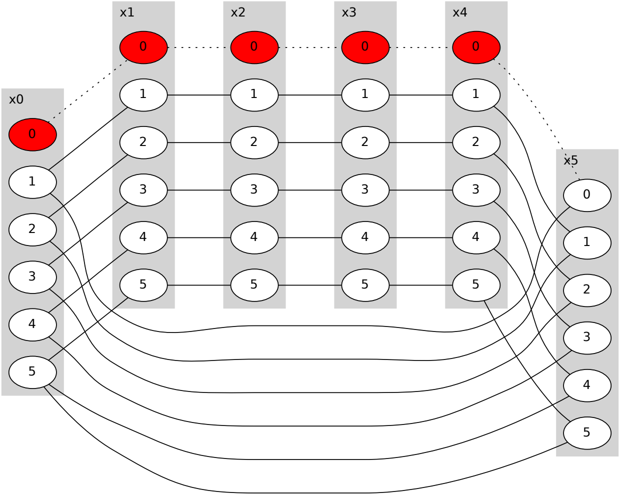

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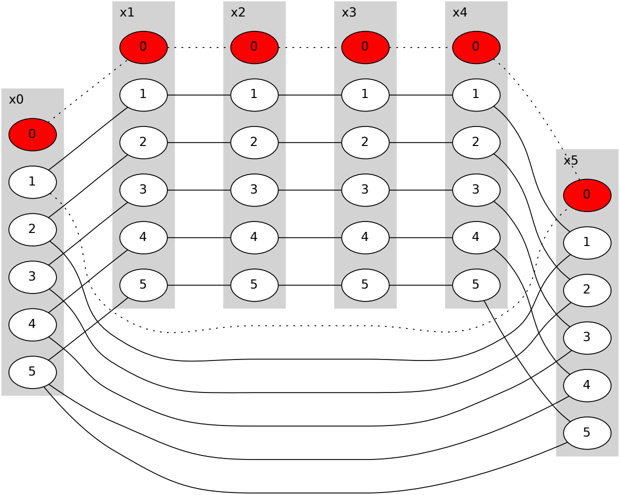
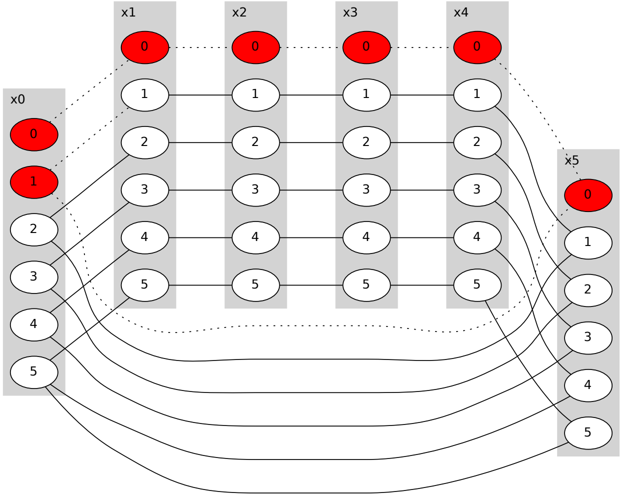
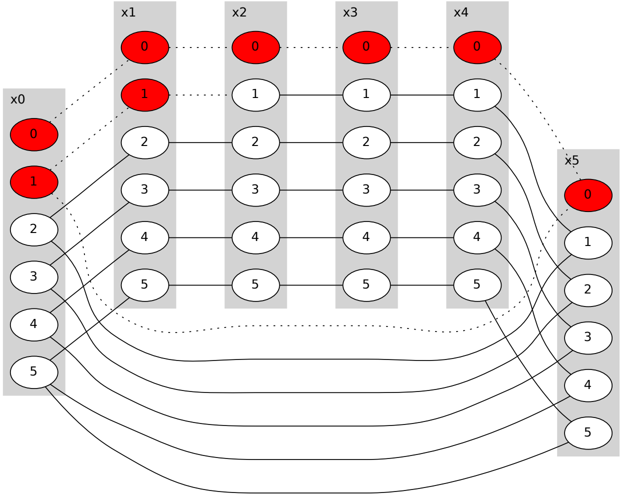


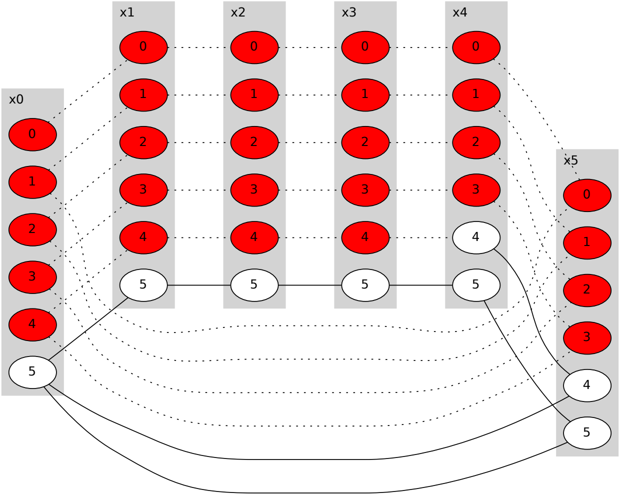
Constraint Propagation on `domino-6`



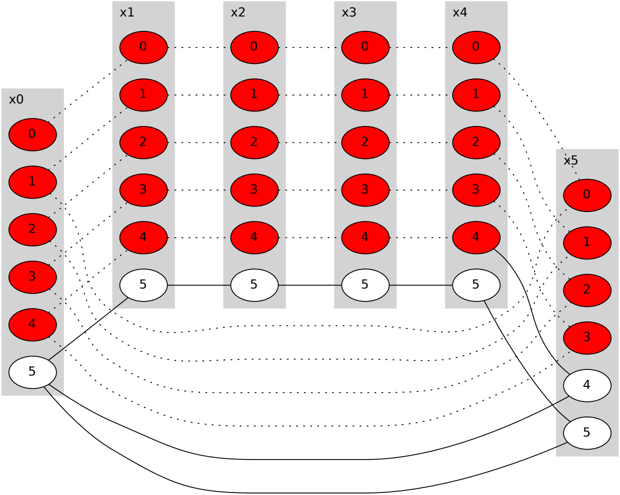
Constraint Propagation on `domino-6`



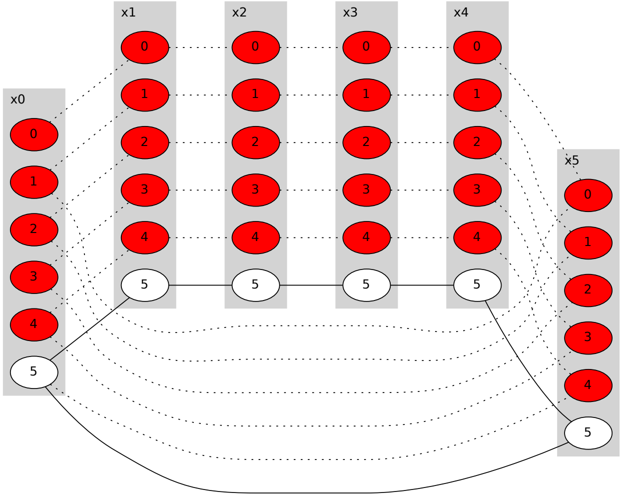
Constraint Propagation on `domino-6`



Constraint Propagation on domino-6



Constraint Propagation on `domino-6`



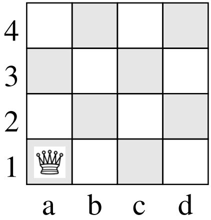
Constraint Propagation on queens-4

For the 4-queens instance, we have:

- $vars(P) = \{$
  x_a with $dom(x_a) = \{1, 2, 3, 4\}$,
  x_b with $dom(x_b) = \{1, 2, 3, 4\}$,
  x_c with $dom(x_c) = \{1, 2, 3, 4\}$,
  x_d with $dom(x_d) = \{1, 2, 3, 4\}$
  $\}$
- $ctrs(P) = \{$
  $x_a \neq x_b \wedge |x_a - x_b| \neq 1$,
  $x_a \neq x_c \wedge |x_a - x_c| \neq 2$,
  $x_a \neq x_d \wedge |x_a - x_d| \neq 3$,
  $x_b \neq x_c \wedge |x_b - x_c| \neq 1$,
  $x_b \neq x_d \wedge |x_b - x_d| \neq 2$,
  $x_c \neq x_d \wedge |x_c - x_d| \neq 1$
  $\}$

Exercice

After taking the decision $x_a = 1$, what is the AC-closure of P?



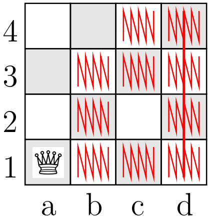
Constraint Propagation on queens-4

For the 4-queens instance, we have:

- $vars(P) = \{$
  x_a with $dom(x_a) = \{1, 2, 3, 4\}$,
  x_b with $dom(x_b) = \{1, 2, 3, 4\}$,
  x_c with $dom(x_c) = \{1, 2, 3, 4\}$,
  x_d with $dom(x_d) = \{1, 2, 3, 4\}$
  $\}$
- $ctrs(P) = \{$
  $x_a \neq x_b \wedge |x_a - x_b| \neq 1$,
  $x_a \neq x_c \wedge |x_a - x_c| \neq 2$,
  $x_a \neq x_d \wedge |x_a - x_d| \neq 3$,
  $x_b \neq x_c \wedge |x_b - x_c| \neq 1$,
  $x_b \neq x_d \wedge |x_b - x_d| \neq 2$,
  $x_c \neq x_d \wedge |x_c - x_d| \neq 1$
  $\}$

Exercice

After taking the decision $x_a = 1$, the AC-closure of P is:



Exercice

Let P be the following CN:

- $vars(P) = \{$
  x_1 with $dom(x_1) = \{1, 2, 3\}$,
  x_2 with $dom(x_2) = \{1, 2, 3\}$,
  x_3 with $dom(x_3) = \{1, 2, 3\}$,
  x_4 with $dom(x_4) = \{1, 2, 3\}$
  $\}$
- $ctrs(P) = \{$
  $x_1 \neq x_2$,
  $x_2 + x_3 \leq x_1$,
  $x_2 + x_4 \geq 2 * x_1$,
  $\}$

Simulate the process of constraint propagation on P (that is to say, compute the AC-closure of P).